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On the stiffness characteristics of nonmonolithic elastic structures. Part I. Theory

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Abstract

The paper analyses the effect of dry joints on the stiffness characteristics of elastic structures. Particular attention is paid to cases with frictional contact sliding because of the indefiniteness of the solution. For this reason a generalized friction law is introduced, where also the displacement discontinuities at the joints are subjected to conical restraints. This law permits a separation of the dissipative component ρ and an auxiliary nondissipative dilatational component β of the friction angle φ. An analysis based on purely nondissipative friction provides unique solutions and thus a framework for the estimate of solutions corresponding to dissipative friction. The main emphasis is laid upon assessment of bounds for the stiffness characteristics of structures. This constitutes an elastic counterpart and complement to an analogous treatment of the stability of rigid body assemblages [Int. J. Solids Struct. 32 (2) (1995) 203].

Keywords: Contact mechanics; Friction; Masonry structures

1. Introduction

The present paper analyses by variational methods the stiffness characteristics of nonmonolithic elastic structures with dry joints based on the linear theory of elasticity. This topic provides an intermediate link between the elastic monolithic structure and the corresponding rigid body assemblage connected by dry joints (Parland, 1995). If contact sliding with friction occurs at the joints, the solution of the static problem is not unique. Variational methods provide suitable means to reduce this indeterminateness. The functionals subjected to variation are generally energy-expressions with appropriate modifications. The extrema of the functionals provide then the variational tools for attainment of the solution of the boundary value problem. We resort to the fact that by direct methods good approximations of the extreme value of the functional are much easier attained, than a satisfactory approximation by variational methods of the complete solution. The mechanical significance of the functionals in question is often fuzzy. Therefore functional characteristics, the structural significance of which are clearly perceptible, are of special interest.

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Such characteristics are the cone of stability $E(P)$ of the loads P for rigid body assemblages, the stiffness $D(P)$ for monolithic elastic structures and the collapse loads P_p for ideally plastic structures. Well known upper and lower bound principles have been established for $D(P)$ (Weber, 1942; Parland, 1951) and P_p (Feinberg, 1948; Hill, 1950) of monolithic structures.

In order to narrow the range of the indefiniteness induced by friction in nonmonolithic structures, we introduce a modified friction law where the friction angle φ of the dry joints obeys the linear law (Parland, 1995)

$$\varphi = \rho + \beta; \quad |\tau| \leqslant |\sigma| \tan(\rho + \beta) \tag{1}$$

Here ρ represents the Coulomb or dissipative friction, whereas β represents the nondissipative or geometric friction, caused by the resistance to frictionless contact sliding along the steepest slope ($\tan \beta$) of the asperity. This linear law seems, according to tests, to materialize at incipient contact sliding and very low stress (Schneider, 1976; Hassanzadeh, 1990). Eq. (1) implies that the stress vector $\mathbf{p} = \{\sigma, \boldsymbol{\tau}\}^{\mathrm{T}}$, as well as the displacement discontinuity vector $\boldsymbol{\gamma} = \{\gamma_n, \boldsymbol{\gamma}_t\}^{\mathrm{T}}$ at the joints $\Gamma_{\nu\mu}$ are locally subjected to conical restraints (Fig. 1).

Problems with purely dissipative friction $\rho \neq 0$ (here labelled DFA) have been largely investigated but recent work concerning the analysis with purely geometric friction $\rho = 0$, $\beta > 0$ (labelled GFA) is scarce—Parland (1968, 1988), Michalowski and Mroz (1978), Sanchez-Palencia and Suquet (1982). The connection of GFA and DFA within a common framework provides, due to the unique characteristics of GFA, bounds to structurally significant stiffness characteristics in DFA. The main purpose of this study is to expound direct methods for the evaluation of these bounds, without resorting to the complete solution of the problem.

In order to distinguish vectors in abstract spaces Y, ∂Y from those in the physical space R^3 we write only the latter with an extra bold letter. Thus $\mathbf{p}(s)$, $\mathbf{u}(s) \in R^3$, but p, $u \in \partial Y$.

2. Mechanics of contact with dry joints, conical restraints

We consider an elastic structure resting on a rigid surface Γ_0 and occupying a domain $\Omega \subset R^3$ with external boundary Γ_e and contact interfaces $\Gamma_{\mu\nu}$. Every $\Gamma_{\mu\nu}$ has a smooth middle-surface $\Gamma^0_{\mu\nu}$ with surface coordinates $s = \{s^1, s^2\}^{\mathrm{T}}$, position vectors $\mathbf{r}^0(s)$, and continuous periodical corrugations $z(s)$ with piecewise continuous integrable gradient ∇z (Fig. 1). We assume that in the initial state the position vectors $\mathbf{r}_\mu(s) \in \Gamma_{\mu\nu}$ and $\mathbf{r}_\nu(s) \in \Gamma_{\nu\mu}$ of opposite faces $\Gamma_{\mu\nu}$ and $\Gamma_{\nu\mu}$ coincide

$$\mathbf{r}_\mu(s) = \mathbf{r}_\nu(s) = \mathbf{r}(s) = \mathbf{r}^0(s) + z(s)\mathbf{n}(s) \tag{2a}$$

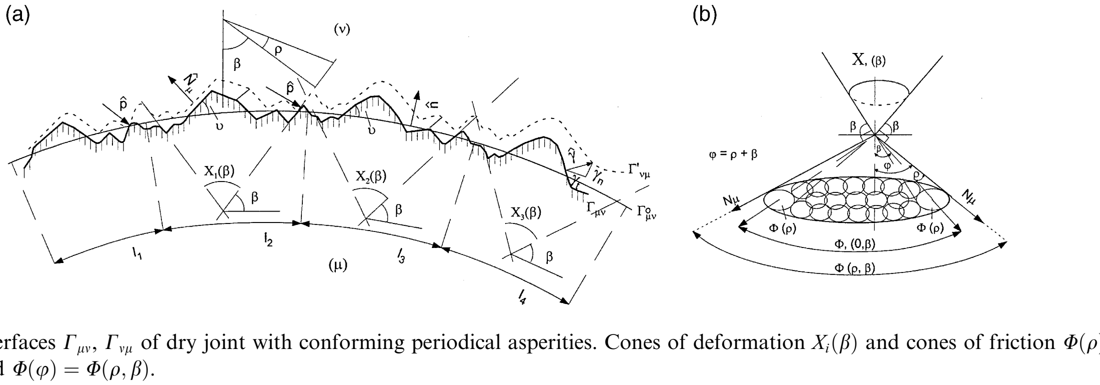

Fig. 1. Interfaces $\Gamma_{\mu\nu}$, $\Gamma_{\nu\mu}$ of dry joint with conforming periodical asperities. Cones of deformation $X_i(\beta)$ and cones of friction $\Phi(\rho)$, $\Phi(0, \beta)$ and $\Phi(\varphi) = \Phi(\rho, \beta)$.

where $\mathbf{n}(s) = \mathbf{n}_\mu^0(s) = -\mathbf{n}_v^0(s)$ is the outside normal of $\Gamma_{\mu v}^0$ of part (μ) and we assume that the system (s^1, s^2, s^3) has orthogonal unit basevectors

$$\mathbf{a}_\alpha = \mathbf{r}_{,\alpha}^0; \quad (\alpha = 1,2); \quad \mathbf{a}_3 = \mathbf{a}_1 \times \mathbf{a}_2 = \mathbf{n}; \quad |\mathbf{a}_\alpha| = 1; \quad \mathbf{a}_1 \cdot \mathbf{a}_2 = 0 \tag{2b}$$

The comma denotes the partial derivative $(\cdot)_{,\alpha} = \partial(\cdot)/\partial s_\alpha$. The length l_i and the roughness amplitude $|z_{\max}|_i$ of a period $\Delta\Gamma_i \subset \Gamma_{\mu v}$ are assumed to be small compared with the linear dimensions L of the structure

$$l_i = \mathrm{O}(\delta L); \quad |z_{\max}|_i = \mathrm{O}(\delta L); \qquad \delta \ll 1 \tag{3a}$$

Furthermore we assume that the curvature of $\Gamma_{\mu v}^0$ and of its coordinate curves, respectively, are of order $1/L$.

$$|\mathbf{n}_{,\beta}| = \mathrm{O}(1/L); \quad |\mathbf{a}_{\alpha,\beta}| = \mathrm{O}(1/L) \qquad (\alpha, \beta = 1,2) \tag{3b}$$

The outside normal $\mathbf{N}_\mu(s)$ of $\mathrm{d}\Gamma_{\mu v}$ within $\Delta\Gamma_i$ is, taking into account $\mathbf{r}_{,\alpha} = (\mathbf{a}_\alpha + z_{,\alpha}\mathbf{n} + z\mathbf{n}_{,\alpha})$, conditions (3a) and (3b) and grad $z = \nabla z = z_{,\alpha}\mathbf{a}_\alpha$

$$\mathbf{N}_\mu(s) = (\mathbf{r}_{,1} \times \mathbf{r}_{,2})/|\mathbf{r}_{,1} \times \mathbf{r}_{,2}| \cong (\mathbf{n}(s) - \nabla z)/(1 + (\nabla z)^2)^{1/2} \tag{4}$$

The inclination $\tan \upsilon_\mathrm{t}(s) = \mathrm{d}z/\mathrm{d}s$ of $\mathrm{d}\Gamma_{\mu v}$ in direction $\mathbf{t} = \mathrm{d}\mathbf{r}^0/\mathrm{d}s$, where $|\mathbf{t}| = 1$, and the maximum inclination $\tan \upsilon$ are defined by

$$\tan \upsilon_\mathrm{t}(s) = \mathbf{t}(s) \cdot \nabla z(s) = |\nabla z| \cos(\nabla z, \mathbf{t}); \quad |\tan \upsilon(s)| = |\nabla z| \tag{5}$$

The continuity of $z(s)$ requires that for any pair s, $s' \in \Delta\Gamma_i$ there holds

$$\mathbf{n}(s') \cdot \mathbf{N}_\mu(s) = \cos \upsilon(s) > 0; \quad \forall s', s \in \Delta\Gamma_i \tag{6}$$

The tangent vector of $\mathrm{d}\Gamma_{\mu v}$ in direction $\mathbf{t}$ is

$$\mathbf{T}_\mu(s) = (\mathbf{t}(s) + \tan \upsilon_\mathrm{t}(s)\mathbf{n}(s)) \cos \upsilon_\mathrm{t}(s); \quad \cos \upsilon_\mathrm{t}(s) = (1 + (\tan \upsilon_\mathrm{t}(s))^2)^{-1/2}; \quad |\mathbf{T}_\mu(s)| = 1 \tag{7}$$

We shall assume throughout that the displacements u, the strains ε_{ij} and the rotations ω_{ij} are infinitesimal, so that all the conditions of the classical linear theory of elasticity hold

$$u = \mathrm{O}(\delta L); \quad \varepsilon_{ij} = \mathrm{O}(\delta); \quad \omega_{ij} = \mathrm{O}(\delta); \quad u_{i,\alpha} = \mathrm{O}(\delta); \quad (\delta \ll 1) \tag{8}$$

The discontinuity of the displacement field across $\Gamma_{\mu v}$ is

$$[\mathbf{u}]_{v\mu}(s) = \mathbf{u}_v(s) - \mathbf{u}_\mu(s) = -[\mathbf{u}]_{\mu v}(s) \tag{9}$$

The vector $[\mathbf{u}]_{v\mu} = \boldsymbol{\gamma}_{v\mu}$ defines uniquely the deformation of the gap between parts (μ) and (v)

$$\boldsymbol{\gamma}_{v\mu}(s) = \boldsymbol{\gamma}_\mathrm{t}(s) + \gamma_\mathrm{n}(s)\mathbf{n}(s); \quad \boldsymbol{\gamma}_\mathrm{t}(s) = \gamma_{\mathrm{t}\alpha}(s)\mathbf{a}_\alpha(s) \tag{10}$$

where $\boldsymbol{\gamma}_\mathrm{t}$ represents the sliding along $\Gamma_{\mu v}^0$ and γ_n the dilatation in the joint.

The impenetrability condition: There is no interpenetration between parts (v) and (μ) on $\Gamma_{\mu v}$. The gap vector between points of the opposite faces induced by $\mathbf{u}$ within a period $\Delta\Gamma_i$ is

$$\mathbf{g}(s, \Delta s) = \mathbf{r}_v'(s + \Delta s) - \mathbf{r}_\mu'(s) = \mathbf{g}_\mathrm{t}(s, \Delta s) + g_\mathrm{n}(s, \Delta s)\mathbf{n}(s) \tag{11a}$$

where $\mathbf{r}_\lambda'(s) = \mathbf{r}_\lambda(s) + \mathbf{u}_\lambda(s)$ for $\lambda = v$, μ. Using the notation $f(s + \Delta s) - f(s) = \Delta f$ we obtain from Eqs. (2a) and (11a) with $\Delta\mathbf{r}_v = \mathbf{r}_v(s + \Delta s) - \mathbf{r}_v(s)$

$$\mathbf{g}(s, \Delta s) = \Delta\mathbf{r}_v + [\mathbf{u}]_{v\mu} + \Delta\mathbf{u}_v = \Delta\mathbf{r}^0 + \Delta z\,\mathbf{n} + z\,\Delta\mathbf{n} + \Delta z\,\Delta\mathbf{n} + \boldsymbol{\gamma}_{v\mu} + \Delta\mathbf{u}_v \tag{11b}$$

Because of conditions (3a) and (8), neglecting quantities $\mathrm{O}(\delta^2 L)$, we obtain from Eqs. (11a) and (11b)

$$\mathbf{g}_\mathrm{t}(s, \Delta s) = \Delta\mathbf{r}^0 + \boldsymbol{\gamma}_\mathrm{t} \tag{11c}$$

$$g_n(s, \Delta s) = \Delta z + \gamma_n \tag{11d}$$

where $\boldsymbol{\gamma}_{\nu\mu}(s)$ is considered to be constant within $\Delta\Gamma_i$ because $|\Delta\mathbf{u}_\nu| = O(\delta^2 L)$. The impenetrability can then be expressed by the condition that for any s and the corresponding Δs, for which $\mathbf{g}_t(s, \Delta s) = 0$, the normal component $g_n(s, \Delta s)$ is nonnegative

$$g_n(s, \Delta s) \geqslant 0; \quad \forall s \in \Delta\Gamma_i \text{ and } \Delta s \text{ for which } \Delta\mathbf{r}^0 = -\boldsymbol{\gamma}_t \tag{12}$$

Hence $\Delta z = \nabla z \cdot \Delta\mathbf{r}^0 = -\nabla z \cdot \boldsymbol{\gamma}_t$, because $\boldsymbol{\gamma}$ is infinitesimal. Therefore the impenetrability condition is expressed by

$$g_n(s) = \gamma_n - \nabla z(\mathbf{s}) \cdot \boldsymbol{\gamma}_t = \gamma_n - |\boldsymbol{\gamma}_t| \tan \upsilon_\gamma(s) \geqslant 0; \quad \forall s \in \Delta\Gamma_i \tag{13}$$

For a given direction $\boldsymbol{\gamma}_t$ condition (13) must be valid for every $s \in \Delta\Gamma_i$, thus

$$\gamma_n \geqslant |\boldsymbol{\gamma}_t| \sup_{s\in\Delta\Gamma_i} (\tan \upsilon_\gamma(s)) = |\boldsymbol{\gamma}_t| \tan \beta_\gamma. \tag{14}$$

where $\tan \beta_\gamma > 0$ is the maximum inclination in direction $\boldsymbol{\gamma}_t$ on $\Delta\Gamma_i$.

Let all $\boldsymbol{\gamma}$ vectors start from a common origin $\mathbf{r}_\gamma^0 \in \Delta\Gamma_i^0$, then

Proposition 1. *The set $\{\gamma\}$ of admissible gap-deformations $\boldsymbol{\gamma}$ in a period $\Delta\Gamma_i \subset \Gamma_{\mu\nu}$ is a closed convex cone $\boldsymbol{X}_i(\beta) \in R^3$, with apex at $\boldsymbol{r}_i^0 \in \Delta\Gamma_i^0$ and lateral surface $\partial\boldsymbol{X}_i$, which is determined by the maximum ascent* $\tan \beta_\gamma$ *in direction $\boldsymbol{\gamma}_t$.*

Proof. $\mathbf{X}_i(\beta)$ is a closed cone because of Eq. (13). For any fixed $s \in \Delta\Gamma_i^0$, Eq. (13) represents a half-space of $\boldsymbol{\gamma}$-vectors bounded by a plane through the origin and parallel to $\mathrm{d}\Gamma(s)$. The set $\{\gamma\}$, that satisfies Eq. (13) for every $s \in \Delta\Gamma_i$, is therefore the intersection of convex sets and therefore forms a convex cone. Contact sliding is excluded for any s with $\tan \upsilon_\gamma(s) < \tan \beta_\gamma$. □

The stress transference at the joint is determined by the equilibrium condition, the no-tension stress condition on $\Gamma_{\mu\nu}^0$ and the friction law. Denoting the stressvector acting on $\mathrm{d}\Gamma_{\mu\nu}$ by $\mathbf{p}_{\mu\nu}^\mu(s)$ and the vector acting on $\mathrm{d}\Gamma_{\nu\mu}$ by $\mathbf{p}_{\nu\mu}^\nu(s)$, there applies at contact points $s, s' \in \Delta\Gamma_i$

$$\mathbf{p}_{\mu\nu}^\mu(s)\,\mathrm{d}\Gamma_{\mu\nu}(s) = -\mathbf{p}_{\nu\mu}^\nu(s')\,\mathrm{d}\Gamma_{\nu\mu}(s') \tag{15}$$

where $\mathbf{p}_{\mu\nu}^\mu$, defined in the local system by basevectors $\mathbf{N}_\mu(s)$ and $\mathbf{T}_\mu(s)$, is subjected to the friction law of Coulomb expressed by the normal stress σ_μ and the shear stress $\boldsymbol{\tau}_\mu$ on $\mathrm{d}\Gamma_{\mu\nu}(s)$

$$\mathbf{p}_{\mu\nu}^\mu = -|\sigma_\mu|\mathbf{N}_\mu + \tau_\mu \mathbf{T}_\mu; \quad \sigma_\mu \leqslant 0 \tag{16a}$$

$$-\tan \rho_\mu^- \leqslant \tau_\mu/|\sigma_\mu| \leqslant \tan \rho_\mu^+ \tag{16b}$$

where the negative lower bound corresponds to reloading on $\mathrm{d}\Gamma_{\mu\nu}$. The corresponding traction $\mathbf{p}_{\mu\nu}$ on $\mathrm{d}\Gamma_{\mu\nu}^0 = \mathrm{d}\Gamma \cos \upsilon$ can be expressed by the base vectors $\mathbf{a}_\alpha$ and $\mathbf{n}$ and corresponding stresses σ and $\boldsymbol{\tau}$

$$\mathbf{p}_{\mu\nu} = -|\sigma|\mathbf{n} + \boldsymbol{\tau}; \quad \boldsymbol{\tau} = \tau_\alpha \mathbf{a}_\alpha; \;\; \mathbf{p}_{\mu\nu} = -\mathbf{p}_{\nu\mu} \tag{17a}$$

Using the relation $\mathbf{p}_{\mu\nu}^\mu(s)\,\mathrm{d}\Gamma_{\mu\nu}(s) = \mathbf{p}_{\mu\nu}(s)\,\mathrm{d}\Gamma_{\mu\nu}^0(s)$ and the Eqs. (4), (5), (7), the friction law (16b), expressed in the base system $\{\mathbf{a}_\alpha, \mathbf{n}\}$ on $\Gamma_{\mu\nu}^0$ for a given direction $\boldsymbol{\tau}$ on $\mathrm{d}\Gamma_{\mu\nu}^0(s)$, renders a generalization of Schneiders friction law

$$-\tan(\rho^- - \upsilon^+) \leqslant \boldsymbol{\tau}(s)/|\sigma(s)| \leqslant \tan(\rho^+ + \upsilon^+) \leqslant \infty; \quad s \in \mathrm{d}\Gamma_{\mu\nu}^0 \tag{17b}$$

where $\tan \upsilon_\tau = (\nabla z \cdot \boldsymbol{\tau})/|\boldsymbol{\tau}|$ and $\tan \rho_\tau = |\tan \rho_\mu| \cos \upsilon / \cos \upsilon_\tau$ with $\cos \upsilon_\tau = (1 + ((\nabla z \cdot \boldsymbol{\tau})/|\boldsymbol{\tau}|)^2)^{-1/2}$. Eq. (17b) includes the no-tension condition in $\Delta\Gamma_i^0 \subset \Gamma_{\mu\nu}^0$ for any $\mathbf{p}_{\mu\nu}(s) = -\mathbf{p}_{\nu\mu}(s)$

$$\sigma(s) = \mathbf{n}(s') \cdot \mathbf{p}_{\mu\nu}(s) < 0; \quad \forall s', s \in \Delta\Gamma_i \tag{18a}$$

Because of Eq. (16b) every $\mathbf{p}^{\mu}_{\mu\nu}(s)$ and $\mathbf{p}_{\mu\nu}(s)$ on $\mathrm{d}\Gamma^0_{\mu\nu}$ are contained in the cone of dissipative friction $\mathbf{\Phi}_\mu(\rho_\mu, s)$, that is assumed to be convex. $\mathbf{\Phi}_\mu(\rho_\mu, s)$ includes the inside normal $-\mathbf{N}_\mu(s)$ of $\mathrm{d}\Gamma_{\mu\nu}$ and since every $\mathbf{p}_{\mu\nu}(s)$ on $\Delta\Gamma_i$ satisfies inequality (18a), the sum of all $\mathbf{\Phi}_\mu(\rho_\mu, s)$ forms a convex cone (Fig. 1b) in every period $\Delta\Gamma^0_i$ with apex at fixed $s_i \in \Delta\Gamma^0_i$

$$\mathbf{\Phi}_i(\varphi) = \sum_k \mathbf{\Phi}_\mu(\rho_\mu, s); \quad \varphi \leqslant \rho_\tau + \beta_\tau \leqslant \pi/2 \tag{18b}$$

This cone, labelled $\mathbf{\Phi}_i(\rho, \beta)$, constitutes the greatest set with max $\varphi^+ = (\rho^+_\tau + \beta^+_\tau)$ and min $\varphi^- = -(\rho^-_\tau + \beta^-_\tau)$

$$\begin{aligned}\mathbf{\Phi}_i(\rho, \beta) = \{&\mathbf{p}_{\mu\nu}(s) = -|\sigma(s)|\mathbf{n}(s) + \boldsymbol{\tau}(s)\mathbf{t}(s);\\ &s \in \Delta\Gamma^0_{\mu\nu}; \quad \sigma(s) \leqslant 0; \quad -\tan\varphi^- \leqslant \tau/|\sigma| \leqslant \tan\varphi^+; \qquad \forall s \in \Delta\Gamma^0_i\}\end{aligned} \tag{19}$$

If $\mathbf{p}_{\mu\nu} \neq 0$, the inequality $|\tau| < |\sigma|\tan\varphi$ defines the interior $\mathbf{\Phi}^0_i(\varphi)$ of $\mathbf{\Phi}_i(\varphi)$, whereas the corresponding equality determines the lateral surface $\partial\mathbf{\Phi}_i(\varphi)$ of $\mathbf{\Phi}_i(\varphi)$. The inequality $\gamma_\mathrm{n} > |\boldsymbol{\gamma}_\mathrm{t}|\tan\beta_\gamma$ defines the interior $\mathbf{X}^0_i(\beta)$ and the equality $\gamma_\mathrm{n} = |\boldsymbol{\gamma}_\mathrm{t}|\tan\beta_\gamma$ defines the lateral surface $\partial\mathbf{X}_i(\beta)$ of $\mathbf{X}_i(\beta)$. In the limit when $\Delta\Gamma_i \to 0$; $\mathrm{z}(s) \to 0$ the sets of cones $\mathbf{X}_i(\beta)$ and $\mathbf{\Phi}_i(\varphi)$ are transformed into sets of cones $\mathbf{X}(\beta, s)$ and $\mathbf{\Phi}(\varphi, s)$, respectively, of the same shape in R^3 at every point $\mathbf{r}^0(s)$ of $\Gamma^0_{\mu\nu}$.

Because dynamic contact requires geometric contact there applies

Correspondence rule. *If $\boldsymbol{p}_{\mu\nu} \in \boldsymbol{\Phi}(\varphi, s)$ and $\boldsymbol{\gamma}_{\nu\mu} \in \boldsymbol{X}(\beta, s)$ correspond to contact, then*:

(α) *nonzero $\boldsymbol{p}_{\mu\nu}(s) \in \boldsymbol{\Phi}^0(\varphi, s)$ implies $\gamma_{\nu\mu}(s) = 0$; complete contact,*
(β) *nonzero $\gamma_{\nu\mu}(s) \in \boldsymbol{X}^0(\beta, s)$ implies $\boldsymbol{p}_{\mu\nu}(s) = 0$; no kinematic and no dynamic contact*
(δ) *at linear contact sliding the nonzero vectors $\boldsymbol{p}_{\mu\nu}(s)$ and $\boldsymbol{\gamma}_{\nu\mu}(s)$ constitute corresponding generatrices of $\partial\boldsymbol{\Phi}(s)$ and $\partial\boldsymbol{X}(s)$ respectively. To these vectors $\boldsymbol{p}_{\mu\nu}(s)$ and $\boldsymbol{\gamma}_{\nu\mu}(s)$ and to any admissible vectors $\boldsymbol{p}''_{\mu\nu}(s) \in \boldsymbol{\Phi}(\varphi, s)$ and $\boldsymbol{\gamma}'_{\nu\mu}(s) \in \boldsymbol{X}(\beta, s)$ there applies the sectional normality rule*:

$$\boldsymbol{\tau} \cdot (\boldsymbol{\gamma}_\mathrm{t}/|\gamma_\mathrm{n}| - \boldsymbol{\gamma}'_\mathrm{t}/|\gamma'_\mathrm{n}|) \geqslant 0; \quad \boldsymbol{\gamma}_\mathrm{t} \cdot (\boldsymbol{\tau}/|\sigma| - \boldsymbol{\tau}''/|\sigma''|) \geqslant 0 \tag{20}$$

Using the notations $\varphi''_\tau = \varphi''$; $\tan\varphi''_{\gamma'} = \tan\varphi''\cos(\boldsymbol{\tau}'', \boldsymbol{\gamma}'_\mathrm{t})$ there follows from Eq. (20)

$$\tan\varphi_\gamma \geqslant \tan\varphi''_\gamma \quad \forall \mathbf{p}''_{\mu\nu} \in \mathbf{\Phi}(\varphi, s) \tag{21a}$$

$$\tan\varphi_\gamma / \tan\beta_\gamma \geqslant \tan\varphi_{\gamma'} / \tan\beta_{\gamma'} \quad \forall \boldsymbol{\gamma}' \in \mathbf{X}(\beta) \tag{21b}$$

The scalar product of admissible $\mathbf{p}''_{\mu\nu}$ and $\boldsymbol{\gamma}'_{\nu\mu}$ can, according to Eq. (19), be written

$$\mathbf{p}''_{\mu\nu}(s) \cdot \boldsymbol{\gamma}'_{\nu\mu}(s) = \boldsymbol{\tau}'' \cdot \boldsymbol{\gamma}'_\mathrm{t} - |\sigma''|\gamma'_\mathrm{n} = |\sigma''||\boldsymbol{\gamma}'_\mathrm{t}|(|\boldsymbol{\tau}''/\sigma''|\cos(\boldsymbol{\tau}'', \boldsymbol{\gamma}'_\mathrm{t}) - \gamma'_\mathrm{n}/|\boldsymbol{\gamma}'_\mathrm{t}|) \tag{22}$$

Because $|\boldsymbol{\tau}''| \leqslant |\sigma''|\tan\varphi''$; $\gamma'_\mathrm{n} \geqslant |\boldsymbol{\gamma}'_\mathrm{t}|\tan\beta'_{\gamma'}$, we obtain

$$\mathbf{p}''_{\mu\nu}(s) \cdot \boldsymbol{\gamma}'_{\nu\mu}(s) \leqslant |\sigma''||\boldsymbol{\gamma}'_\mathrm{t}|(\tan\varphi''_{\gamma'} - \tan\beta'_{\gamma'}) \tag{23}$$

Because of Correspondence rule (δ), there holds

$$\sup_{\gamma'} \frac{\mathbf{p}_{\mu\nu} \cdot \boldsymbol{\gamma}'_{\nu\mu}}{|\sigma||\boldsymbol{\gamma}'_\mathrm{t}|} = \sup_{\mathbf{p}''} \frac{\mathbf{p}''_{\mu\nu} \cdot \boldsymbol{\gamma}_{\nu\mu}}{|\sigma''||\boldsymbol{\gamma}_\mathrm{t}|} = \frac{\mathbf{p}_{\mu\nu} \cdot \boldsymbol{\gamma}_{\nu\mu}}{|\sigma||\boldsymbol{\gamma}_\mathrm{t}|} = (\tan\varphi_\gamma - \tan\beta_\gamma) \tag{24a}$$

Thus if $\mathbf{p}_{\mu\nu}(s)$ and $\boldsymbol{\gamma}_{\nu\mu}(s)$ are corresponding vectors, then

$$\mathbf{p}_{\mu\nu}(s) \cdot \boldsymbol{\gamma}_{\nu\mu}(s) = |\sigma||\boldsymbol{\gamma}_\mathrm{t}|(\tan\varphi_\gamma - \tan\beta_\gamma) \geqslant 0 \tag{24b}$$

If $[\mathbf{r}]_{\nu\mu} = h\mathbf{n}$, where $h \geqslant 0$, impenetrability requires with $\gamma_n + h \geqslant |\boldsymbol{\gamma}_t| \tan\beta_\gamma$

$$\mathbf{p}''_{\mu\nu} \cdot \boldsymbol{\gamma}'_{\nu\mu} \leqslant |\sigma''|(|\boldsymbol{\gamma}'_t|(\tan\varphi''_{\gamma'} - \tan\beta_{\gamma'}) + h) \tag{25a}$$

$$\mathbf{p}_{\mu\nu} \cdot \boldsymbol{\gamma}_{\nu\mu} = |\sigma|(|\boldsymbol{\gamma}_t|(\tan\varphi_\gamma - \tan\beta_\gamma) + h) \tag{25b}$$

3. General characteristics of nonmonolithic structures

Consider a possible state of equilibrium (PE) with the governing equations for the state of stress $\{\sigma_{ij}\}$ and the loads $f(\Omega)$ and $p(\Gamma_c)$

$$\sigma_{ij,i} + f_j = 0; \quad \sigma_{ij} = \sigma_{ji} \qquad i,j = 1,2,3 \text{ in } \Omega \tag{26a}$$

$$\sigma_{ij} n_i = p_j \quad \text{on } \Gamma_e \tag{26b}$$

$$(\sigma_{ij} n_i)_\mu = p_{\mu\nu j}; \quad (\sigma_{ij} n_i)_\nu = p_{\nu\mu j} \quad \text{on } \Gamma^0_{\mu\nu}, \Gamma^0_{\nu\mu} \tag{27a}$$

$$p_{\mu\nu j}(s) = -p_{\nu\mu j}(s) \quad \text{on } \Gamma_c = \sum \Gamma^0_{\mu\nu} \tag{27b}$$

This possible state is an admissible equilibrium state (AE) if p on Γ_e satisfies certain nonhomogeneous loading conditions (Section 4) and the friction condition (19) on $\Gamma_c = \sum \Gamma^0_{\mu\nu}$

$$p(\Gamma_c) \in \Phi(\rho, \beta, \cdot); \quad \Phi(\rho, \beta, \cdot) = \cup \boldsymbol{\Phi}(\rho, \beta, \cdot) \tag{27c}$$

A possible kinematic state (PK) satisfies the conditions

$$\varepsilon_{ij} = 1/2(u_{i,j} + u_{j,i}) \tag{28a}$$

$$[\mathbf{u}] = \mathbf{u}_\nu(s) - \mathbf{u}_\mu(s) = \boldsymbol{\gamma}_{\nu\mu}(s) \quad \text{on } \Gamma_{\mu\nu} \tag{28b}$$

$$\mathbf{u}_0 = \mathbf{0} \quad \text{on } \Gamma_0 \tag{28c}$$

This possible state is an admissible kinematic state (AK) if the displacements on Γ_e satisfy certain nonhomogeneous kinematic conditions and the impenetrability condition (14b) on Γ_c

$$\gamma(\cdot) \in X(\beta, \cdot); \quad X(\beta, \cdot) = \cup \mathbf{X}(\beta, \cdot) \tag{28d}$$

The symmetric tensor E_{ijrs} and its inverse E^{-1}_{ijrs} connect stresses and strains by the relations

$$\sigma_{ij} = E_{ijrs} \varepsilon_{rs}; \quad \varepsilon_{ij} = E^{-1}_{ijrs} \sigma_{rs} \tag{29}$$

These equations together with the relations (27a)–(28c) provide the means for the solution of the displacement problem.

Considering the possible states $\mathrm{PE}'' = \{p''(\Gamma_e), f''(\Omega), \sigma''(\Omega), p''(\Gamma_c)\}$ and $\mathrm{PK}' = \{u', \varepsilon', \gamma'\}$ we obtain from the multiplication of Eq. (26a) by u' and using Gauss–Green's theorem

$$\int_\Omega (\sigma''_{ij,i} + f''_j) u'_j \,\mathrm{d}\Omega = \int_{\Gamma_e} \sigma''_{ij} n_i u'_j \,\mathrm{d}\Gamma + \sum \left(\int_{\Gamma_{\mu\nu}} (\sigma''_{ij} n_i)_\mu u'_{j\mu} \,\mathrm{d}\Gamma + \int_{\Gamma_{\nu\mu}} (\sigma''_{ij} n_i)_\nu u'_{j\nu} \,\mathrm{d}\Gamma \right) - \int_\Omega \sigma''_{ij} u'_{j,i} \,\mathrm{d}\Omega + \int_\Omega f''_j u'_j \,\mathrm{d}\Omega = 0 \tag{30}$$

Recalling Eqs. (27a)–(27c) and (28b) and combining opposite traction vectors on Γ_c we get the virtual work relation between internal and external work

$$\int_{\Omega} \sigma''_{ij}\varepsilon'_{ij}\,\mathrm{d}\Omega + \int_{\Gamma_c} \mathbf{p}''_{\mu\nu}\cdot\boldsymbol{\gamma}'_{\nu\mu}\,\mathrm{d}\Gamma = \int_{\Omega} \mathbf{f}''\cdot\mathbf{u}'\,\mathrm{d}\Omega + \int_{\Gamma_e} \mathbf{p}''\cdot\mathbf{u}'\,\mathrm{d}\Gamma \tag{31}$$

Referring to Romano and Sacco (1985) we introduce on Ω, $\Gamma_c = \sum \Gamma^0_{\mu\nu}$ and Γ_e inner product spaces and their dual spaces:

(a) Y the space of int. loads $f(\Omega)$
(b) ∂Y the space of surface loads $p(\Gamma_e)$ on Γ_e
(c) H space of stress $\sigma_{ij}(\Omega)$
(d) ∂H space of joint stresses $p(\Gamma_c)$ including reactions $p_{\mu 0}$ on $\sum \Gamma^0_{\mu 0}$
(e) $\overline{Y} = Y \oplus \partial Y$ space of force loads $\bar{p} = \{f(\Omega), p(\Gamma_e)\}^{\mathrm{T}}$
(f) $\overline{H} = H \oplus \partial H$ space of internal forces $\bar{\sigma} = \{\sigma(\Omega), p(\Gamma_c)\}^{\mathrm{T}}$

(a′) Y' space including displacements $u(\Omega)$
(b′) $\partial Y'$ space including surface displacements $u(\Gamma_e)$ on Γ_e
(c′) H' space including strains $\varepsilon_{ij}(\Omega)$
(d′) $\partial H'$ space including joint deformations $\gamma(\Gamma_c)$ and $\gamma_{\mu\nu}$ on $\sum \Gamma^0_{\mu 0}$
(e′) $\overline{Y}' = Y' \oplus \partial Y'$ space incl. displacements $\bar{u} = \{u(\Omega), u(\Gamma_e)\}^{\mathrm{T}}$
(f′) $\overline{H}' = H' \oplus \partial H'$ space incl. deformations $\bar{\varepsilon} = \{\varepsilon(\Omega), \gamma(\Gamma_c)\}^{\mathrm{T}}$

The scalar product and the norm in H are defined by the bilinear form $c(\sigma, \sigma'')$

$$(\sigma|\sigma'')_H = c(\sigma,\sigma'') = \int_{\Omega} E^{-1}_{ijrs}\sigma_{ij}\sigma''_{rs}\,\mathrm{d}\Omega \tag{32a}$$

$$\|\sigma''\| = c(\sigma'',\sigma'')^{1/2} = \left(\int_{\Omega} E^{-1}_{ijrs}\sigma''_{ij}\sigma''_{rs}\,\mathrm{d}\Omega\right)^{1/2} = (2W''_{\sigma})^{1/2} \tag{32b}$$

The corresponding quantities in H' for kinematically possible $\varepsilon(u)$, $\varepsilon(u')$ are defined by the bilinear form $e(\varepsilon(u), \varepsilon(u')) = a(u, u')$

$$(\varepsilon|\varepsilon')_{H'} = a(u,u') = \int_{\Omega} E_{ijrs}\varepsilon_{ij}\varepsilon'_{rs}\,\mathrm{d}\Omega \tag{33a}$$

$$\|\varepsilon'\| = a(u',u') = \left(\int_{\Omega} E_{ijrs}\varepsilon'_{ij}\varepsilon'_{rs}\,\mathrm{d}\Omega\right)^{1/2} = (2W'_{\varepsilon})^{1/2} \tag{33b}$$

W_σ denotes the stress energy and W_ε the strain energy of the structure. The bilinear forms $c(\sigma, \sigma'')$ and $a(u, u')$ are symmetric and positive definite which satisfy Schwarz's inequalities

$$c(\sigma'',\sigma'')\cdot c(\sigma,\sigma) \geqslant c(\sigma'',\sigma)^2; \quad a(u',u')\cdot a(u,u) \geqslant a(u',u)^2. \tag{34}$$

If $\{\sigma''\} \subset \mathrm{PE}$ and $\{u'\} \subset \mathrm{PK}$ there follows from Eqs. (31)–(33b) and Schwarz's inequality

$$\int_{\Omega} \sigma''_{ij}\varepsilon'_{ij}\,\mathrm{d}\Omega = c(\sigma'',\sigma(u')) = a(u(\sigma''),u') \leqslant (c(\sigma'',\sigma'')\cdot a(u',u'))^{1/2} = (4W''_{\sigma}\cdot W'_{\varepsilon})^{1/2} \tag{35}$$

Since the admissible vectors $\mathbf{p}(s)$ and $\boldsymbol{\gamma}(s)$ on $\Delta\Gamma_i$ constitute convex cones $\boldsymbol{\Phi}(\varphi, s)$ and $\mathbf{X}(\beta, s)$ respectively, the set of admissible functions $p(\cdot) \in \partial H$, $\gamma(\cdot) \in \partial H'$ constitute convex cones $\Phi(\varphi, \cdot) \subset \partial H$ and $X(\beta, \cdot) \subset \partial H'$, respectively. The integrals in Eq. (31) can be expressed by dual pairings in H, $\overline{H}$ and Y, $\overline{Y}$, respectively, and the work equality (31) can be written as

$$\langle \sigma'', \varepsilon'\rangle_H + \langle p'', \gamma'\rangle_{\partial H} = \langle f'', u'\rangle_Y + \langle p'', u'\rangle_{\partial Y} \tag{36a}$$

or

$$\langle \bar{\sigma}'', \bar{\varepsilon}'\rangle_{\overline{H}} = \langle \bar{p}'', \bar{u}'\rangle_{\overline{Y}} \tag{36b}$$

where

$$\langle \sigma'', \varepsilon' \rangle_H = \int_\Omega \sigma''_{ij} \varepsilon'_{ij} \, d\Omega; \; \langle p'', \gamma' \rangle_{\partial H} = \int_{\Gamma_c} \mathbf{p}''_{\mu\nu} \cdot \boldsymbol{\gamma}'_{\nu\mu} d\Gamma; \; \langle f'', u' \rangle_Y = \int_\Omega \mathbf{f}'' \cdot \mathbf{u}' \, d\Omega; \; \langle p'', u' \rangle_{\partial Y} = \int_{\Gamma_e} \mathbf{p}'' \cdot \mathbf{u}' \, d\Gamma$$

4. The loading conditions and the stiffness of nonmonolithic structures

The loading conditions are usually expressed by prescribed loads p^* on a part Γ_p^* of the external surface Γ_e, where u^0 is unspecified, and by prescribed nonzero displacements u^* on $\Gamma_u^* \subset \Gamma_e$, where p^0 is unspecified. In this case, for a solution $\{u, \sigma\}$ with corresponding states $\{\sigma, p(\Gamma_c)\}$, $\{u, \varepsilon, \gamma\}$, where according to Eqs. (32a), (32b), (33a) and (33b) $\langle \sigma, \varepsilon \rangle = 2W_\varepsilon = 2W_\sigma$, the Eq. (36a) can be written as

$$2W + \langle p, \gamma \rangle_{\partial H} = \langle f^*, u \rangle_Y + \langle p^*, u^0 \rangle_{\partial Y} + \langle p^0, u^* \rangle_{\partial Y} \tag{37}$$

In contact problems the parts of surfaces, where loads p and displacements u are given, cannot generally be separated. Thus in the indentation problem of a rigid stamp into an elastic layer the resultant force R and the gradient of the displacement are simultaneously prescribed. In this case the loading can be expressed in the spaces ∂Y, Z and their duals $\partial Y'$, Z' by (Fig. 2)

$$\mathrm{B}p = P^*; \; \mathrm{B} : \partial Y \to Z; \; \mathrm{C}'u = U^*; \; \mathrm{C}' : \partial Y' \to Z' \tag{38}$$

where Z and Z' are the spaces generated by the scalar field of coordinates P_i and U_i of the functions $p(\cdot) \in \partial Y$ and $u(\cdot) \in \partial Y'$, respectively. B and C′ and their adjoints B′ and C are bounded linear operators with ranges R(B) and R(C′), respectively. Using the decomposition

$$p = p^* + p^0; \; (p^*|p^0)_{\partial Y} = 0; \; \mathrm{B}p^0 = 0; \; p \in \partial Y \tag{39}$$

$$u = u^* + u^0; \; (u^*|u^0)_{\partial Y'} = 0; \; \mathrm{C}'u^0 = 0; \; u \in \partial Y' \tag{40}$$

the set $\{u^0\}$ constitutes the nullspace $N(\mathrm{C}')$ of C′ and the set $\{p^0\}$ constitutes the null-space $N(\mathrm{B})$ of B. The set $\{p^*\} = \partial Y_\mathrm{B}^*$ is $N(\mathrm{B})$'s orthogonal complement $N(\mathrm{B})^\perp$ in ∂Y. Hence $\partial Y = \partial Y_\mathrm{B}^* \oplus N(\mathrm{B})$ and analogously $\partial Y' = \partial Y_\mathrm{C}'^* \oplus N(\mathrm{C}')$. Conditions (39) and (40) represent for fixed P^*, U^* linear varieties M^* and M'^* in ∂Y and $\partial Y'$, respectively, generated by the translated subspaces $N(\mathrm{B})$ and $N(\mathrm{C}')$. Because $N(\mathrm{B})$ and $N(\mathrm{C}')$ are

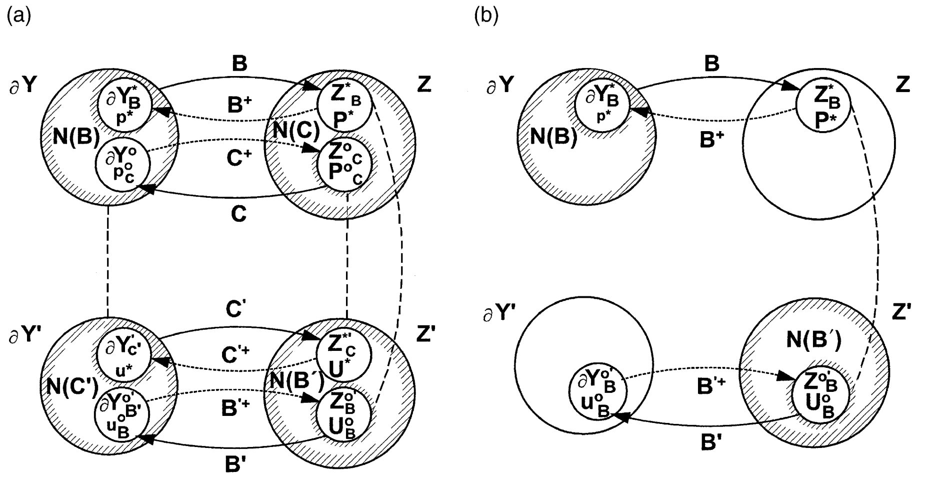

Fig. 2. Scheme of loadings. (a) Loads p^* and displacements u^* prescribed. (b) Only loads p^* prescribed.

closed and $p^* \perp N(\mathrm{B})$, $u^* \perp N(\mathrm{C}')$, p^* and u^* represent the perpendiculars from the origin to the translated $N(\mathrm{B})$ and $N(\mathrm{C}')$ respectively. Hence, if $\mathrm{B}p = P^*$ and $\mathrm{C}'u = U^*$ on Γ^*, then

$$M^* = p^* + N(\mathrm{B}); \quad \|p^*\| = \min\|p\|; \qquad M'^* = u^* + N(\mathrm{C}'); \quad \|u^*\| = \min\|u\| \tag{41}$$

Definition. The loading conditions are said to be *complementary* if p^*, u^* and p^0, u^0 on Γ^* satisfy

$$\langle p^*, u^*\rangle_{\partial Y} = 0; \quad \langle p^0, u^0\rangle_{\partial Y} = 0 \tag{42}$$

Complementarity requires that the subsets $\{p^0_{\mathrm{C}}\} \subset N(\mathrm{C}')^{\perp} = \mathrm{C}P^0$ and $\{u^0_{\mathrm{B}}\} \subset N(\mathrm{B})^{\perp} = \mathrm{B}'U^0$ of $\{p^0\}$ and $\{u^0\}$, respectively (Fig. 2), are orthogonal, $\{p^0_{\mathrm{C}}\} \perp \{u^0_{\mathrm{B}}\}$.

Let $u^0_{\mathrm{B}} \in N(\mathrm{C}')$ be the component of u^0 orthogonal to $N(\mathrm{B})$, and $p^0_{\mathrm{C}} \in N(\mathrm{B})$ be the component of p^0 orthogonal to $N(\mathrm{C}')$, then there applies (Luenberger, 1968):

Lemma 1. *If the loading conditions on Γ^* are complementary ($\langle p^*, u^*\rangle_{\partial Y} = 0$, $\langle p^0, u^0\rangle_{\partial Y} = 0$) then*:

(*a*)

$$\{p^*\} = \{B^+P^*\} = \partial Y^*_B; \quad \{u^*\} = \{\mathrm{C}'^+U^*\} = \partial Y_{C'^*} \tag{43}$$

where B^+ and C'^+ are pseudoinverse operators $B^+ : Z_B \to \partial Y^*_B$; $C'^+ : Z'_C \to \partial Y_{C'^*}$ with inverses $(B^+)^{-1} = B$ and $(C'^+)^{-1} = C'$

(*b*)

$$\text{If } N(B) \neq 0, \quad \text{then } \{u^0_B\} = \{B'U^0\} = R(B'); U^0 \in Z'_B \tag{44a}$$

(*c*)

$$\text{If } N(C') \neq 0 \quad \text{then } \{p^0_C\} = \{CP^0\} = R(C); P^0 \in Z_C \tag{44b}$$

(*d*) *The work on Γ_e is*

$$\langle p, u\rangle_{\partial Y} = \langle p^*, u^0_B\rangle_{\partial Y} + \langle p^0_C, u^*\rangle_{\partial Y} \tag{44c}$$

(*e*) *If $N(C')$ and $N(B)$ are nonempty then BC and $C'B'$ are null-operators*

Proof. Since $\mathrm{B} : \partial Y^*_{\mathrm{B}} \to Z_{\mathrm{B}}$ and $\mathrm{C}' : \partial Y_{\mathrm{C}'^*} \to Z'_{\mathrm{C}}$ are one to one and onto, there is a one to one correspondence between p^* and P^* and u^* and U^*, respectively. Therefore corresponding pseudoinverse operators $\mathrm{B}^+ : Z_{\mathrm{B}} \to \partial Y^*_{\mathrm{B}}$; $\mathrm{C}'^+ : Z'_{\mathrm{C}} \to \partial Y_{\mathrm{C}'^*}$ exist. Since ∂Y and $\partial Y'$ are inner product spaces, any subspace $\partial Y_{\mathrm{d}} \subset \partial Y$ can be identified with its dual $(\partial Y_{\mathrm{d}})' \subset \partial Y'$ and vice versa. Because $\{p^*\} \perp \{p^0\} \in N(\mathrm{B})$ and $\{u^*\} \perp \{u^0\} \in N(\mathrm{C}')$ there follows

$$\{p^*\} \in \partial Y^*_{\mathrm{B}} = (\partial Y_{\mathrm{B}})' = N(\mathrm{B})^{\perp} = R(\mathrm{B}'); \quad \{u^*\} \in \partial Y'^*_{\mathrm{C}} = \partial Y_{\mathrm{C}} = N(\mathrm{C}')^{\perp} = R(\mathrm{C}) \tag{45a}$$

The condition $\langle p^0, u^0\rangle_{\partial Y} = 0$ implies $\langle p^0, u^0_{\mathrm{B}}\rangle_{\partial Y} = 0$, $\langle p^0_{\mathrm{C}}, u^0\rangle_{\partial Y} = 0$ and $\langle p^0_{\mathrm{C}}, u^0_{\mathrm{B}}\rangle_{\partial Y} = 0$. Hence

$$p^0_{\mathrm{C}} \in N(\mathrm{C}')^{\perp} = R(\mathrm{C}); \quad u^0_{\mathrm{B}} \in N(\mathrm{B})^{\perp} = R(\mathrm{B}'), \tag{45b}$$

from which follows (b)–(d): $\langle p, u\rangle_{\partial Y} = \langle p^* + p^0, u^* + u^0\rangle_{\partial Y} = \langle p^*, u^0_{\mathrm{B}}\rangle_{\partial Y} + \langle p^0_{\mathrm{C}}, u^*\rangle$. Therefore $\langle p^0_{\mathrm{C}}, u^0_{\mathrm{B}}\rangle_{\partial Y} = \langle \mathrm{C}P^0, \mathrm{B}'U^0\rangle_Z = \langle \mathrm{BC}P^0, U^0\rangle_Z = \langle P^0, \mathrm{C}'\mathrm{B}'U^0\rangle_Z = 0$, from which follows (e). □

The direction of the operators can be reversed for instance $\mathrm{C}' : Z' \to \partial Y'$ (Appendix A). If B or C′ are identity operators from ∂Y and $\partial Y'$, their nullspaces are empty. On account of Lemma 1, the external work on Γ_{e} can be expressed in spaces Z, Z' by

$$\langle p, u\rangle_{\partial Y} = \langle p^*, \mathrm{B}'U^0\rangle_{\partial Y} + \langle \mathrm{C}P^0, u^*\rangle_{\partial Y} = \langle P^*, U^0\rangle_Z + \langle P^0, U^*\rangle_Z \tag{46}$$

If complementarity holds, the work equation (37) for a solution at load $\{f^*, p^*, u^*\}$ is

$$2W + \langle p, \gamma \rangle_{\partial H} = \langle f^*, u \rangle_Y + \langle p^*, u_B^0 \rangle_{\partial Y} + \langle p_C^0, u^* \rangle_{\partial Y} \tag{47a}$$

where u_B^0 and p_C^0 are the components of u^0 and p^0, orthogonal to any p^0 and any u^0, respectively. It retains in these more complicated cases the same form as in Eq. (37). Expressing (47a) in spaces $\overline{H}$ and $\overline{Y}$ and using the notation $\bar{m}^* = \bar{p}^*/\|\bar{p}^*\|$ we get, if $u^* = 0$,

$$\langle \bar{\sigma}, \bar{\varepsilon} \rangle_{\overline{H}} = \langle \bar{p}^*, \bar{u}_B \rangle_{\overline{Y}} = \|\bar{p}^*\| \langle \bar{m}^*, \bar{u}_B \rangle_{\overline{Y}}; \quad (\|\bar{p}^*\|^2 = \|f^*\|^2 + \|p^*\|^2) \tag{47b}$$

This detailed analysis of the loading conditions is unavoidable for the definition of the stiffness characteristics of nonmonolithic structures. In these, the stiffness cannot be defined by flexibility or stiffness matrices, but must be based on more general methods. Thus the stiffness for the point load $\mathbf{P}$ is defined as the ratio of $|\mathbf{P}|$ to the load–displacement U_p in the direction of $\mathbf{P}$

$$D = |\mathbf{P}|/U_p = |\mathbf{P}|^2/(\mathbf{P} \cdot \mathbf{U})$$

In the general case with $u^* = 0$ the load perpendicular $\bar{p}^* \in \overline{Y}$ defines the stiffness by

$$D = \|\bar{p}^*\| / \langle \bar{m}^*, \bar{u}_B \rangle_{\overline{Y}} = \|\bar{p}^*\|^2 / \langle \bar{p}^*, \bar{u}_B \rangle_{\overline{Y}}; \quad \bar{m}^* = \bar{p}^*/\|\bar{p}^*\| \tag{48a}$$

Using Eq. (47b), where the internal work depends on the friction ρ, β the stiffness can be expressed alternatively by

$$D_\sigma(\rho, \beta) = \frac{\|\bar{p}^*\|^2}{\langle \bar{\sigma}, \bar{\varepsilon} \rangle_{\overline{H}}}; \quad D_\varepsilon(\rho, \beta) = \frac{\langle \bar{\sigma}, \bar{\varepsilon} \rangle_{\overline{H}'}}{\langle \bar{m}^*, \bar{u}_B \rangle^2_{\overline{Y}'}} \tag{48b}$$

because $\|\bar{p}^*\| = \langle \bar{\sigma}, \bar{\varepsilon} \rangle_{\overline{H}} / \langle \bar{m}^*, \bar{u}_B \rangle_{\overline{Y}}$. The minimum norm $\|\bar{p}^*\|$ expresses the load intensity. At given $\bar{p}^*$ the norm $\|\bar{p}^*\|$ in expression (47b) can be replaced by the norm of any component or linear transformation of $\bar{p}^*$.

Proposition 2 (Multiplicity rule). *Let us assume that in the initial state of the structure there are no eigenstresses σ_0 and no initial gaps $[r](\cdot)$ on the interfaces. If the loading increases proportionally from zero and a solution $\{u, \sigma\}$ corresponds to the load $\{f^*, p^*, u^*\}$, then a solution $\{\lambda u, \lambda \sigma\}$ corresponds to the load $\{\lambda f^*, \lambda p^*, \lambda u^*\}$, where actual contact prevails on unchanged interfaces and the equilibrium remains stable, if and only if $\lambda > 0$.*

Proof. The stable solutions comprises corresponding AE- and AK-states governed by linear relations and conical restraints. The effect of sliding on the contact area can be overlooked because of geometrical linearity with $u = O(\delta L)$. Hence the "if" part is obvious. $\lambda < 0$ is excluded because of the conical restraints. □

If the correspondence rule holds, stiffnesses defined by Eq. (48b) are independent of the load intensity λ.

5. Characteristics of the solution according to nondissipative friction

In this case the friction angles on the interfaces are $\rho = 0$ and $\varphi = \beta$. The scalar product of any $\mathbf{p}''_{\mu\nu} \in \mathbf{\Phi}_i(0, \beta)$ and any $\boldsymbol{\gamma}'_{\nu\mu} = \gamma'_n \mathbf{n} + \boldsymbol{\gamma}'_t \in \mathbf{X}_i(\beta)$ is according to Eq. (22) and the inequalities $(|\boldsymbol{\tau}''|/|\sigma''|)\cos(\boldsymbol{\tau}'', \boldsymbol{\gamma}'_t) \leqslant \tan\beta_{\gamma'}$ and $\gamma'_n \geqslant |\boldsymbol{\gamma}'_t| \tan\beta_{\gamma'}$

$$\mathbf{p}''_{\mu\nu} \cdot \boldsymbol{\gamma}'_{\nu\mu} = |\sigma''||\boldsymbol{\gamma}'_t| \left(\frac{\boldsymbol{\tau}'' \cdot \boldsymbol{\gamma}'_t}{|\sigma''||\boldsymbol{\gamma}'_t|} - \frac{\gamma'_n}{|\boldsymbol{\gamma}'_t|} \right) \leqslant |\sigma''||\boldsymbol{\gamma}'_t| \left[\max\left((|\boldsymbol{\tau}''|/|\sigma''|) \cos(\boldsymbol{\tau}'', \boldsymbol{\gamma}'_t) \right) - \min\left(\gamma'_n / |\boldsymbol{\gamma}'_t| \right) \right] \leqslant 0 \tag{49}$$

Equality holds according to Eq. (24b) only for the corresponding vectors $\mathbf{p}_{\mu\nu}$, $\boldsymbol{\gamma}_{\nu\mu}$

$$\mathbf{p}_{\mu\nu}(s) \cdot \boldsymbol{\gamma}_{\nu\mu}(s) = 0 \tag{50}$$

In the spaces ∂H and $\partial H'$ the formulae (49) and (50) correspond to the dual pairings

$$\langle p''(\cdot), \gamma(\cdot)\rangle_{\partial H} \leqslant 0; \quad \langle p(\cdot), \gamma(\cdot)\rangle_{\partial H} = 0 \tag{51}$$

From Eqs. (49)–(51) there follows

Lemma 2.

(*i*) *The friction cone $\boldsymbol{\Phi}_i(0,\beta)$ is the negative normal cone $\boldsymbol{X}_i^*(\beta)$ of the cone $\boldsymbol{X}_i(\beta)$ of admissible gap deformations in $\Delta\Gamma_i : \boldsymbol{\Phi}_i(0,\beta) = \boldsymbol{X}_i^*(\beta) \subset R^3$.*

(*ii*) *The corresponding cone $\Phi(0,\beta,\cdot) \subset \partial H$ is the negative normal cone $X^*(\beta,\cdot)$ of $X(\beta,\cdot) \subset \partial H'$.*

(*iii*) *Corresponding nonzero vectors* $p(\cdot)$ and $\gamma(\cdot)$ *are orthogonal generatrices of $\partial\Phi(0,\beta,\cdot)$ and $\partial X(\beta,\cdot)$, respectively* (*Correspondence rule* δ).

Theorem 1. *If the friction is nondissipative* ($\varphi = \beta$), *the boundary conditions are complementary and a solution exists, that corresponds to initial stress σ_0, initial gaps $[r]$ and external load p^*, this solution is unique.*

Proof. (a) If at load $\{p^*, u^*\}$ there would be two solutions $\{u^1, \sigma^1\}$ and $\{u^2, \sigma^2\}$, then their difference $\{u^2 - u^1, \sigma^2 - \sigma^1\}$ would according to Eq. (38) satisfy the condition

$$\langle \sigma^2 - \sigma^1, \varepsilon^2 - \varepsilon^1\rangle_H + \langle p^2 - p^1, \gamma^2 - \gamma^1\rangle_{\partial H} = \langle p^2 - p^1, u^2 - u^1\rangle_{\partial Y} \tag{52a}$$

where $p^i(\Gamma_e) = p^{*i} + p^{0i}$; $u^i = u^{*i} + u^{0i}$; $i = (1,2)$. Therefore, since $p^{*2} - p^{*1}$, $u^{*2} - u^{*1} = 0$ and $\langle p^{0i}, u^{0j}\rangle_{\partial Y} = 0$, there applies because of complementarity

$$\langle \sigma^2 - \sigma^1, \varepsilon^2 - \varepsilon^1\rangle_H + \langle p^2 - p^1, \gamma^2 - \gamma^1\rangle_{\partial H} = \langle p^{02} - p^{01}, u^{02} - u^{01}\rangle_{\partial Y} = 0 \tag{52b}$$

The first term on the left hand side equals $a(u^2 - u^1, u^2 - u^1)$, which is independent of σ_0, $[r]$ and is positive definite. The second term is independent of σ_0 and $[r]$, because $\gamma^2 - \gamma^1 = ([r] + \gamma^2) - ([r] + \gamma^1)$. Recalling Eq. (51) and $[r] + \gamma^i \in X(\beta)$ with $\langle p^i, [r] + \gamma^i\rangle = 0$ $(i = 1,2)$, this leads to

$$(\langle p^2, [r] + \gamma^2\rangle_{\partial H} + \langle p^1, [r] + \gamma^1\rangle_{\partial H}) - (\langle p^1, [r] + \gamma^2\rangle_{\partial H} + \langle p^2, [r] + \gamma^1\rangle_{\partial H}) \geqslant 0 \tag{53}$$

Hence the left hand side of Eq. (52b) is non-negative and vanishes only if all differences $\{u^2 - u^1, \sigma^2 - \sigma^1\}$ vanish. On the contrary, noncomplementarity may imply several solutions. □

5.1. Stiffness characteristics

Let u^*, $[r]$, $\sigma_0 = 0$ and let $W''_\sigma(\varphi)$ denote the stress energies of the states $\{p^*, \sigma'', p''(\Gamma_c) \in \Phi(\varphi,\cdot)\}$, then $W''_\sigma(\varphi)$, where $\varphi = \rho + \beta$, includes also energies $W_\sigma(0,\varphi)$ of the nondissipative states $\{u, \sigma\}_{0,\varphi}$.

Lemma 3. *The stress energy $W''_\sigma(\varphi)$ of a AE-state $\{p^*, \sigma'', p''(\Gamma_c)\}_\varphi$, corresponding to the load p^* and the given friction cone $\Phi(\varphi,\cdot)$, attains its minimum at a solution $\{u,\sigma\}_{0,\varphi}$ that exists if and only if this solution corresponds to conical nondissipative friction $\varphi = \beta$, $\rho = 0$*

$$W''_\sigma(\varphi) \geqslant W_\sigma(0,\varphi); \quad \gamma(\cdot) \in X(\varphi,\cdot) = \Phi^*(\varphi,\cdot) \tag{54}$$

Proof.

(a) The "if" part is proved by assuming that the solution $\{u,\sigma\}$ corresponds to nondissipative friction $\varphi = \beta$. If complementarity holds, $\langle p^0, u^0\rangle = 0$, the work equations for PE-states $\{0, \sigma'' - \sigma, p'' - p\}$ and the solution $\{u, \sigma\}$ can, according to Eqs. (34) and (35), be written

$$\langle \sigma'' - \sigma, \varepsilon \rangle = c(\sigma'' - \sigma, \sigma) = -\langle p'' - p, \gamma \rangle_{\partial H} \geqslant 0 \tag{55}$$

because $\langle p, \gamma \rangle_{\partial H} = 0$, $\langle p'', \gamma \rangle_{\partial H} \leqslant 0$. Recalling Schwarz's inequality we obtain $c(\sigma'', \sigma'') \geqslant c(\sigma, \sigma)$ from which follows $W_\sigma(0, \varphi) \leqslant W''_\sigma(\varphi)$.

(b) The "only if" statement is proved by assuming $W_\sigma(\varphi) = \min W''_\sigma(\varphi)$, where $\{\sigma\}$ minimizes $W''_\sigma(\varphi)$ and simultaneously belongs to a solution of the contact problem at load p^*. The original minimization problem is

$$\text{find } \{\sigma, p(\Gamma_c)\} \text{ such that } c(\sigma, \sigma) \leqslant c(\sigma'', \sigma'') \quad \text{with } p(\Gamma_c) \in \Phi(\varphi, \cdot) \text{ and } \{\sigma''\} \in \text{AE} \tag{56}$$

The weak formulation of Eq. (56) is (Ekeland and Temam, 1974)

$$\text{find } \{\sigma, p(\Gamma_c)\} \text{ such that } c(\sigma'' - \sigma, \sigma) \geqslant 0 \quad \text{with } p(\Gamma_c) \in \Phi(\varphi, \cdot) \tag{57a}$$

The solution $\{u, \sigma\}$ comprises a AK-state $\{u, \varepsilon, \gamma\}$, where the constraints on γ are unspecified. Because $c(\sigma, \sigma'')$ is symmetric and taking into account Eqs. (35), (55) and (57a) we obtain

$$c(\sigma'' - \sigma, \sigma) = \langle \sigma'' - \sigma, \varepsilon \rangle_H = -\langle p'' - p, \gamma \rangle_{\partial H} \geqslant 0; \quad \forall\, p''(\cdot) \in \Phi(\varphi, \cdot) \subset \partial H \tag{57b}$$

If we assume that interface $\Gamma_{\mu\nu}$ has fixed parts, we can choose such an AE-state $\{p^*, \sigma''(\Omega), p''(\Gamma_c)\}$ that $p''(\Gamma_c) = p(\Gamma_c)$, except on a measurable set $\{s\} = \mathrm{d}\Gamma^0_{\mu\nu}$ of a detachable Γ_c. Eq. (57b) can then be written

$$(\mathbf{p}'' - \mathbf{p}) \cdot \boldsymbol{\gamma} \leqslant 0 \tag{58a}$$

Choosing $\mathbf{p}'' = \lambda \mathbf{p}$, with $\lambda > 1$ we get $\mathbf{p} \cdot \boldsymbol{\gamma} \leqslant 0$ and with $0 < \lambda < 1$ we get $\mathbf{p} \cdot \boldsymbol{\gamma} \geqslant 0$. Hence for corresponding vectors $\mathbf{p}$ and $\boldsymbol{\gamma}$ there holds

$$\mathbf{p} \cdot \boldsymbol{\gamma} = 0; \quad \mathbf{p} \in \boldsymbol{\Phi}(\varphi) \tag{58b}$$

and for not corresponding $\mathbf{p}''$ and $\boldsymbol{\gamma}'$ there holds

$$\mathbf{p}'' \cdot \boldsymbol{\gamma}' < 0; \quad \forall\, \mathbf{p}'' \in \boldsymbol{\Phi}(\varphi) \tag{58c}$$

The relations (58a)–(58c) can be satisfied only if every $\gamma(\cdot)$ is restricted to a cone $X(\varphi, \cdot) = \Phi^*(\varphi, \cdot)$, the negative normal cone of $\Phi(\varphi, \cdot)$. But this means that the friction is conically nondissipative. □

The lemma is a generalization of Castigliano's principle of minimum stress energy and it connects conical nondissipative friction directly with this principle.

If u^*, $[r]$, $\sigma_0 = 0$ the *stiffness* $D(0, \beta) = \|\bar{p}^*\| / \langle \bar{m}^*, \bar{u} \rangle_{\bar{Y}}$ corresponding to a solution $\{u, \sigma\}_{0,\beta}$ at load $\bar{p}^*$ can, according to Eqs. (48a) and (48b), be expressed with $\bar{m}^* = \bar{p}^* / \|\bar{p}^*\|$ alternatively by

$$D_\varepsilon(0, \beta) = \frac{2W_\varepsilon(0, \beta)}{\langle \bar{m}^*, \bar{u} \rangle^2_{\bar{Y}}} \tag{59a}$$

or

$$D_\sigma(0, \varphi) = \frac{\|\bar{p}^*\|^2}{2W_\sigma(0, \varphi)} \tag{59b}$$

Analogically we introduce the concepts of stiffness $D'_\varepsilon(\beta)$ of varied AK-states $\{u', \varepsilon', \gamma'(\Gamma_c)\}_\beta$ and stiffness $D''_\sigma(\varphi)$ of varied AE-states $\{\sigma'', p''(\Gamma_c)\}_\varphi$ at the same load p^*

$$D'_\varepsilon(\beta) = \frac{2W'_\varepsilon(\beta)}{\langle \bar{m}^*, \bar{u} \rangle^2_{\bar{Y}}} \tag{59c}$$

$$D''_\sigma(\varphi) = \frac{\|\bar{p}^*\|^2}{2W''_\sigma(\varphi)} \tag{59d}$$

The following extremum principles of stiffness are valid for nondissipative friction ($\rho = 0$) in the whole range $0 \leqslant \{\beta, \varphi\} \leqslant \pi/2$ if the structure in the initial state is unstressed ($\sigma_0 = 0$) and at the joints there are no initial gaps ($[r] = 0$).

Theorem 2. *If the structure is subjected to a load $\bar{p}^*$ and the boundary conditions are complementary and with respect to the displacements homogeneous, then the stiffness*

$$D'_{\varepsilon}(\beta) = \frac{2W'_{\varepsilon}(\beta)}{\langle \bar{m}^*, \bar{u}' \rangle^2_{\overline{Y}}} \tag{60a}$$

defined for all kinematically admissible states $\{u', \varepsilon', \gamma'\}_{\beta}$, where $\gamma'(\Gamma_c) \in \mathrm{X}(\beta, \cdot)$ and $\langle \bar{p}^*, \bar{u}' \rangle_{\overline{Y}} > 0$, attains an absolute minimum $D(0, \beta)$ in the actual the actual nondissipative state $\{u, \sigma\}_{0,\beta}$ and the stiffness

$$D''_{\sigma}(\varphi) = \frac{\|\bar{p}^*\|^2}{2W''_{\sigma}(\varphi)} \tag{60b}$$

defined for all admissible equilibrium states $\{p^*, \sigma'', p''\}_{\varphi}$, where $p''(\Gamma_c) \in \Phi(\varphi, \cdot)$, attains an absolute maximum $D(0, \varphi)$ in the actual nondissipative state $\{u, \sigma\}_{0,\varphi}$. Thus

$$\min_{u'} D'_{\varepsilon}(\beta) = D(0, \beta) \tag{61a}$$

$$\max_{\sigma''} D''_{\sigma}(\varphi) = D(0, \varphi) \tag{61b}$$

If $\varphi = \beta$ there holds:

$$D''_{\sigma}(\beta) \leqslant D(0, \beta) \leqslant \mathrm{D}'_{\varepsilon}(\beta) \tag{61c}$$

Proof. The lower bound statement follows immediately from Lemma 2. The upper bound statement follows applying the work equation (34) to the solution $\{u, \sigma\}_{0,\beta}$ and an admissible $\{u', \varepsilon', \gamma'_{\beta}\}$, where $\gamma, \gamma' \in X(\beta, \cdot)$. Recalling Eq. (58c) and $\forall\, \bar{u}' \perp \bar{p}^0$ on Γ^*_{e}, we obtain

$$\langle \sigma(u), \varepsilon(u') \rangle_H = a(u, u') = \langle \bar{p}^* + \bar{p}^0, \bar{u}' \rangle_{\overline{Y}} - \langle p(u), \gamma(u') \rangle_{\partial H} \geqslant \|\bar{p}^*\| \langle \bar{m}^*, \bar{u}' \rangle_{\overline{Y}} \tag{62a}$$

$$\langle \sigma(u), \varepsilon(u) \rangle_H = a(u, u) = \|\bar{p}^*\| \langle \bar{m}^*, \bar{u} \rangle_{\overline{Y}} - \langle p(u), \gamma(u) \rangle_{\partial H} = \|\bar{p}^*\| \langle \bar{m}^*, \bar{u} \rangle_{\overline{Y}} \tag{62b}$$

according to Eq. (51). Dividing Eq. (62a) by $\langle \bar{m}^*, \bar{u}' \rangle_{\overline{Y}}$ and Eq. (62b) by $\langle \bar{m}^*, \bar{u} \rangle_{\overline{Y}}$, subtraction gives $a(u,u)/\langle \bar{m}^*, \bar{u} \rangle \leqslant a(u,u')/\langle \bar{m}^*, \bar{u}' \rangle$. From this and Schwarz's inequality there follows $(a(u,u)/\langle \bar{m}^*, \bar{u} \rangle)^2 \leqslant (a(u,u')/\langle \bar{m}^*, \bar{u}' \rangle)^2 \leqslant (a(u,u) \cdot a(u',u'))/\langle \bar{m}^*, \bar{u}' \rangle^2$. Dividing by $a(u,u)$ and inserting $W'_{\varepsilon}(\beta)$ and $W(0,\beta)$ we obtain condition (61a). $\square$

If p and u on Γ^* are defined in $Z = R^n$ by generalized loads and displacements $P^* = \{P_1 \dots P_n\}^{\mathrm{T}}$, $U^0 = \{U_1 \dots U_n\}^{\mathrm{T}}$, the solution satisfies, by Eqs. (46) and (59a)–(59d), the work equation

$$\langle P^*, U^0 \rangle_Z = \langle \sigma, \varepsilon \rangle_H = 2W_{\sigma}(P^*) \tag{63a}$$

According to the multiplicity rule every U_i and every $\partial W/\partial P_i$ are homogeneous first-degree functions of the P_i, and $W(P)$ is a homogeneous second-degree function of the P_i. Hence

$$\partial W/\partial P_i = \sum_j (\partial^2 W/\partial P_i \partial P_j) P_j, \quad W = \frac{1}{2} \sum_i P_i(\partial W/\partial P_i) = \frac{1}{2} \sum_i \sum_j (\partial^2 W/\partial P_i \partial P_j) P_i P_j \tag{63b}$$

By varying P_i we obtain from Eq. (63a) $\mathrm{d}_i W = \langle \mathrm{d}_i \sigma, \varepsilon \rangle_H = \mathrm{d}P_i U_i - \langle \mathrm{d}_i p, \gamma \rangle_{\partial H}$. If $p \in \partial\Phi(0, \beta, \cdot)$ and $p + \mathrm{d}_i p \in \Phi(0, \beta, \cdot)$ and because $\Phi(0, \beta, \cdot)$ is convex, $\langle \mathrm{d}_i p, \gamma \rangle_{\partial H} \leqslant 0$. Therefore U^0 is a subgradient of W. If all $\partial\Phi(0,\beta,s)$ on Γ_c are smooth, then $d\mathbf{p}\cdot\boldsymbol{\gamma}=0$ because of the correspondence rule. In this case there holds

$$\frac{\partial W}{\partial P_i}=U_i \tag{64a}$$

$$\frac{\partial U_i}{\partial P_j}=\frac{\partial U_j}{\partial P_i} \tag{64b}$$

These equations are generalizations of Castigliano's and Maxwell's rules.

The *stiffness vector* $\Delta=\{\Delta_1\ldots\Delta_n\}^{\mathrm{T}}\in Z$ is defined by

$$\Delta(\rho,\beta)=P^*D^{1/2}/\|P^*\| \tag{65a}$$

$$\|\Delta\|=D^{1/2} \tag{65b}$$

Δ has the direction of the load P^* in Z and defines the *stiffness surface* $F(\Delta,\rho,\beta)$, which encloses the origin of Δ. If the friction is nondissipative, we obtain by substituting $P_i=\|P^*\|\Delta_i/\|\Delta\|$ into Eq. (63b)

$$\frac{1}{D}=\frac{2W(0,\beta)}{\|P\|^2}=\sum_i\sum_j\frac{\partial^2 W}{\partial \mathrm{P}_i\partial P_j}\frac{\Delta_i\Delta_j}{D} \tag{66a}$$

This defines the stiffness surface $F(\Delta,\rho,\beta)$ for $\rho=0$, that can be expressed by

$$F(\Delta,0,\beta)=\sum_i\sum_j\frac{\partial^2 W}{\partial P_i\partial P_j}\Delta_i\Delta_j-1=0 \tag{66b}$$

or

$$2W(\Delta)=1 \tag{66c}$$

Because of Eq. (64a) we obtain

$$U_i=\frac{\partial W(\Delta)}{\partial\Delta_i}\frac{\|P^*\|}{\|\Delta\|} \tag{67a}$$

$$\sum\Delta_iU_i>0 \tag{67b}$$

The outside normal $n_{\mathrm{F}}(\Delta)$ of $F(\Delta,0,\beta)$ has components $n_i(\Delta)=(\partial W(\Delta)/\partial\Delta_i)/\|\partial W(\Delta)/\partial\Delta_i\|_Z$.

Therefore, where $F(\Delta)$ is smooth, we get the normality rule

$$n_{\mathrm{F}}=U/\|U\| \tag{68}$$

At a cornerpoint Δ_c of $F(\Delta,0,\beta)$ the U is contained in the normal cone of $F(\Delta_c,0,\beta)$.

Let $2W(\Delta_1)\leqslant 1$ and $2W(\Delta_2)\leqslant 1$ and since $F(\Delta,0,\beta)=2W(\Delta)-1$, then if $0<\alpha<1$, we get

$$(2W(\alpha\Delta_1+(1-\alpha)\Delta_2))^{1/2}\leqslant\alpha(2W(\Delta_1)^{1/2}+(1-\alpha))(2W(\Delta_2))^{1/2}\leqslant 1 \tag{69}$$

because $(2W(\Delta))^{1/2}=a(\Delta,\Delta)$ can be regarded as a norm of Δ in a transformed Z space.

If the structure is monolithic, the terms $\partial^2W/\partial P_i\partial P_j$ are constant elements of a matrix $[K^{ij}]$.

The expression

$$\mathrm{EM}(\Delta)=\sum\sum K^{ij}\Delta_i\Delta_j-1=0 \tag{70}$$

represents then the *stiffness ellipsoid* EM of the monolithic structure with corresponding stiffness vectors $\Delta(M)$ and stiffnesses $D(M)$. The set of loads $\{P_{\mathrm{k}}\}$, that induces in the nonmonolithic structure the states of stress and strain of the monolithic structure, constitutes the cone $E_k(0,\varphi)$ of the monolithic kern of the structure (Fig. 6). Within $E_k(0,\varphi)$ complete geometrical contact ($\gamma = 0$) and maximum dynamic contact ($p(\Gamma_c) \in \Phi^0(\varphi,\cdot)$) prevail on every detachable surface. $E_k(0,\varphi)$ is convex, because if P_k^1 induces $p_k^1(\Gamma_c) \in \Phi^0(\varphi,\cdot)$ and P_k^2 induces $p_k^2(\Gamma_c) \in \Phi^0(\varphi,\cdot)$ then $p_k^1 + p_k^2 \in \Phi^0(\varphi,\cdot)$, where $\Phi^0(\varphi,\cdot)$ is the interior of $\Phi(\varphi,\cdot)$ that is convex.

From Theorem 1 and formulae (68)–(70) there follows:

Proposition 3.
(i) *The stiffness surface $F(\Delta,0,\beta)$ is uniquely determined by β and is convex*
(ii) *The work $\sum \Delta_i U_i$ is positive, $\sum \Delta_i U_i > 0$*
(iii) *The displacement U is contained in the normal cone $\{n_F\}$ of the stiffness surface $F(\Delta,0,\beta)$*
(iv) *$F(\Delta,0,\beta)$ is enclosed in the stiffness ellipsoid EM of the corresponding monolithic structure. It coincides with the stiffness ellipsoid EM where the load is within the convex cone of the elastic kern $E_k(0,\beta)$.*

Proposition 4. *If the assemblage is detachable, the stiffness surface $F(\Delta,0,\beta)$ approaches asymptotically a generatrix $\partial E(0,\beta)$ of the cone of stability $E(0,\beta)$ of the corresponding rigid body assemblage in the neighbourhood of their common origin θ. If $\Delta \to 0$, the normal n_F of $F(\Delta,0,\beta)$ approaches the normal n_E of $\partial E(0,\beta)$ and U approaches $\partial \Xi$, where $\Xi = E^*(0,\beta)$, the normal cone of $E(0,\beta)$, represents the cone of detachment.*

Proof. If the interfaces $\Gamma_{\mu\nu}$ separate the structure into detachable parts, a load P^c that coincides with a generatrix ∂E of the cone $E(0,\beta)$ induces an unbounded displacement U with zero stiffness $\Delta = 0$. According to Eqs. (67b) $\sum \Delta_j U_i = |P^*|/|\Delta| > 0$. This is possible because of the collinearity of n_F and U only if $U \to \infty$ when $\Delta \to 0$. But unboundedness of U implies that $\Delta \to \partial E(0,\beta)$ and $U \perp \partial E(0,\beta)$ in the neighbourhood of θ. On $\partial E(0,\beta) P^c \perp U^c \in \partial \Xi(\beta)$, therefore $n_E = U^c/\|U^c\| = n_F(0)$ (see Part II, Section 4). □

5.2. Extent of contact and limit state of free contact

If we have a detachable interface $\Gamma_{\mu\nu}$ with friction, we can distinguish three regions:

(a) The stick-region Γ_k where $p(\Gamma_k) \in \Phi^0(\varphi,\Gamma_k)$ and $\gamma(\Gamma_k) = 0$; proper sticking.
(b) The slip-region Γ_s where nonzero $\gamma(\Gamma_s) \in \partial X(\beta,\Gamma_s)$ and corresponds to nonzero $p(\Gamma_s) \in \partial \Phi(\varphi,\Gamma_s)$.
(c) The detachment region Γ_d where nonzero $\gamma(\Gamma_d) \in X^0(\beta,\Gamma_d)$ corresponds to $p(\Gamma_d) = 0$.

If, at given load p^*, we make a very thin cut Γ_1 from outside that induces nondissipative friction along an internal surface where originally the normal stress $\sigma > 0$, this cut will generate a detachment region Γ_d with a stress discontinuity at the tip of Γ_1. If further increase of Γ_1 induces compressive stresses $\sigma < 0$, this may cause contact sliding in a region $\Delta\Gamma_1 = \Gamma_s$ with a stress discontinuity at the tip $\partial\Gamma_k$ of the cut. If there is a border $\partial\Gamma_k$ for the not-cut region Γ_n, across which the joint traction p changes continuously from $p \in \partial\Phi(\varphi,\cdot)$ to the interior $\Phi^0(\varphi,\cdot)$ of the friction cone and the gap deformation γ from outside approaches zero, this border $\partial\Gamma_k^0$ defines the limit state of unconstrained contact. The position of the moving boundary $\partial\Gamma_k^0$ may be determined by parameters $r_1 \dots r_m$ such that Γ_n increases monotonically with r_i.

Proposition 5. *If at given load p^* the friction in the cut interface Γ_c is nondissipative and the thickness t of the cut is minute, there holds*:

(*a*) *The stiffness $D(0,\varphi,r)$ increases monotonously with the not-cut area Γ_n (Fig. 3).*

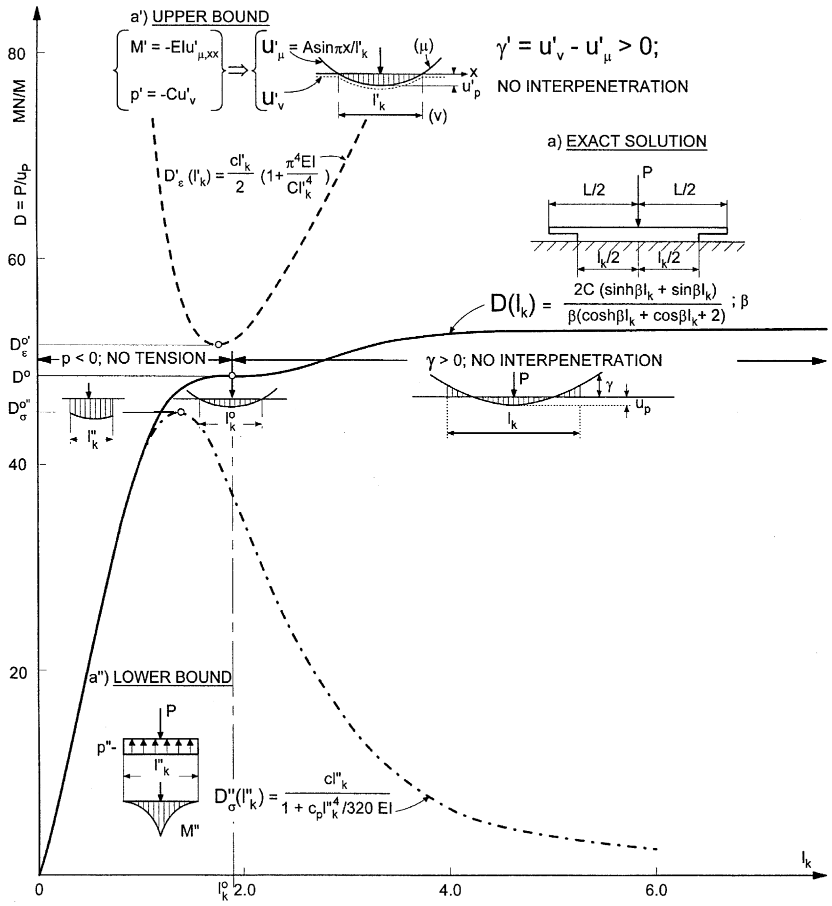

Fig. 3. Beam on Winkler-foundation. Dependence of stiffnesses D, D'_ε and D''_σ on contact length l_k, l'_k and l''_k, respectively.

(*b*) *In the limit state of unrestrained contact the stiffness attains a maximum* $D^0(0,\beta)$ *at values* r_i^0 *within a neighbourhood* $\Delta\Gamma_k$ *of* $\partial\Gamma_n^0$ *where* $p(\Delta\Gamma_k)\in\Phi^0(0,\varphi)$

$$\left(\frac{\partial D}{\partial r_i}\right)_{\Delta\Gamma_\mathrm{k}}=0;\quad \left(\frac{\partial^2 D}{\partial r_i\partial r_j}\right)_{\Delta\Gamma_\mathrm{k}}=0;\qquad \forall\ r_i,r_j\in\Delta\Gamma_\mathrm{k} \tag{71a}$$

(*c*) *The generalized displacements* U_i *attain extreme values in the limit state. This implies*

$$\left(\frac{\partial U_i}{\partial r_j}\right)_{\Delta\Gamma_\mathrm{k}}=0;\quad \forall\ U_i,r_j;\ \ r_j\in\Delta\Gamma_\mathrm{k} \tag{71b}$$

Proof. (a) Because at given load p^* any increase of Γ_n relaxes the restraints on the state of stress $\{\sigma(\Omega),p(\Gamma_\mathrm{c})\}$, this can only increase the stiffness or keep it unchanged, according to the maximum principle of Theorem 2, from which (a) follows. Because on $\partial\Gamma_\mathrm{k}$ a continuous transition occurs from $p_1(\Gamma_1)\in\partial\Phi(0,\varphi)$ to $p(\Gamma_\mathrm{k})\in\Phi^0(0,\varphi)$, there is within $\partial\Gamma_\mathrm{k}$ a region $\Delta\Gamma_\mathrm{k}$, where an additional cut does not affect the state of stress and strain. This means that the state of stress and strain and the stiffness, within an additional cut in region $\Delta\Gamma_k$, remains unchanged. From this follows the disappearance of the first and second variations of $D(r_1, \dots, r_n)$ on $r_i \in \Delta\Gamma_k$ and so condition (b) and also the independence of any U_i on $r_i \in \Delta\Gamma_k$, which gives condition (c). □

In the case $\tan\varphi = 0$, or $\gamma_t(\Gamma_1) = 0$, a cut of finite thickness t reduces Γ_s to zero and the region $\Delta\Gamma_k$ shrinks to a narrow band $\Delta\Gamma^0$ containing $\partial\Gamma^0$. At the limit boundary $\partial\Gamma^0$ the stiffness $D(0, \varphi, r)$ attains an inflexion point with respect to r_i and the generalized displacements $U_i(r)$ attain extreme values (Fig. 3).

Example 1. An elastic beam (μ) loaded by a point load P and resting on a Winkler-foundation (v). The elementary calculations are based on the correspondence rule

$$\gamma_{v\mu} = u_v - u_\mu > 0; \quad p_{\mu v} = 0 \text{ on } \Gamma_d; \quad \gamma_{v\mu} = 0; \quad u_v = u_\mu \neq 0; \quad p_{\mu v} = -cu_\mu \text{ on } \Gamma_k = l_k \tag{72a}$$

with the loading condition on $\Gamma^* = L$ (L total length of beam) and the differential equations

$$P = \int_{\Gamma*} p\,dx; \quad p_{\mu v} = -M_{,xx}; \quad u_{\mu,xxxx} + 4\beta^4 u_\mu = 0; \quad \beta = (c/4EJ)^{1/4} \tag{72b}$$

Applying Proposition 5 to the extremum principles of Theorem 2 we obtain according to Fig. 3.

$$D''_\sigma(1^{0''}) = 1.704c/\beta < D(1^0) = 1.838\,c/\beta < D'_\varepsilon(1^{0'}) = 1.944\,c/\beta \tag{72c}$$

This elementary example shows, that with very simple approximations, the extrema of D'_ε and D''_σ provide acceptable bounds for the stiffness of the limit state (and for this only).

Example 2. Smooth ($\varphi = 0$) eccentrically loaded rigid beam on halfplane. With notations according to Fig. 4 and $e' = e - d/2 + d'/2$ the inclination of the beam θ and the pressure $p(x)$ on the contact area $\Gamma_k = d'$ are (Milne-Thomson, 1960; Heinisuo, 1983):

$$\theta = \frac{2(1+\kappa)}{\mu\pi(d')^2}\text{Pe}' \tag{72d}$$

$$p(x) = \frac{P(1 + e'x/(d')^2)}{\pi((d'/2)^2 - x^2)^{1/2}} \tag{72e}$$

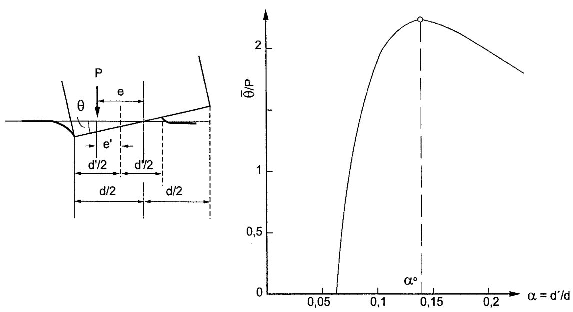

Fig. 4. Indentation of elastic half-plane by a rigid rectangular stamp. Relative inclination ratio $\bar{\theta}/P = (2.66 - 3(1-\alpha))/\alpha^2$ at eccentricity $e = 0.44$d versus contact area ratio $\alpha = d'/d$.

Hence, θ attains a maximum at $d^{0\prime} = 2(d - 2e)$

$$\theta_{\max} = \frac{(1+\kappa)P}{4\mu\pi(d - 2e)} \tag{72f}$$

that corresponds to the limit state of free contact with $p(-d^{0\prime}/2) = 0$.

6. Stiffness characteristics of the solution according to dissipative friction (DFA)

In the general case we have contact sliding with friction angles ρ, $\beta > 0$. The stiffness $D(\rho, \beta)$ of the structure depends entirely on the loading history (Fig. 5b). The uniqueness of the DFA-solution can be established only in special cases (Part II, Appendix B).

If u^*, $[r]$, $\sigma_0 = 0$, let $\{u, \sigma\}_{\rho,\beta}$ be a solution corresponding to a proportional loading and friction angles ρ, β. The gap work is, recalling Eq. (24b)

$$\langle p, \gamma\rangle_{\partial H} = \langle |\sigma|, |\gamma_t|(\tan\varphi_\gamma - \tan\beta_\gamma)\rangle_{\partial H} \geqslant 0 \tag{73}$$

The stiffness $D(\rho, \beta)$ can then be expressed, using Eqs. (26a), (26b), (48a) and (48b), either by the corresponding AK-state $\{u, \varepsilon, \gamma\}_{\rho,\beta}$ as $D_\varepsilon(\rho, \beta)$ with $W_\varepsilon(\rho, \beta)$ and $\bar{m}^* = \bar{p}^*/\|\bar{p}^*\|$, or by the corresponding AE-state $\{\bar{p}^*, \sigma(\Omega), p(\Gamma_c)\}_{\rho,\beta}$ as $D_\sigma(\rho, \beta)$ with $W_\sigma(\rho, \beta)$

$$D_\varepsilon(\rho, \beta) = \frac{2W_\varepsilon(\rho, \beta) + \langle p(u), \gamma\rangle_{\partial H}}{\langle m^*, u\rangle^2} \tag{74a}$$

$$D_\sigma(\rho, \beta) = \frac{\|p^*\|^2}{2W_\sigma(\rho, \beta) + \langle p, \gamma(\sigma)\rangle_{\partial H}} \tag{74b}$$

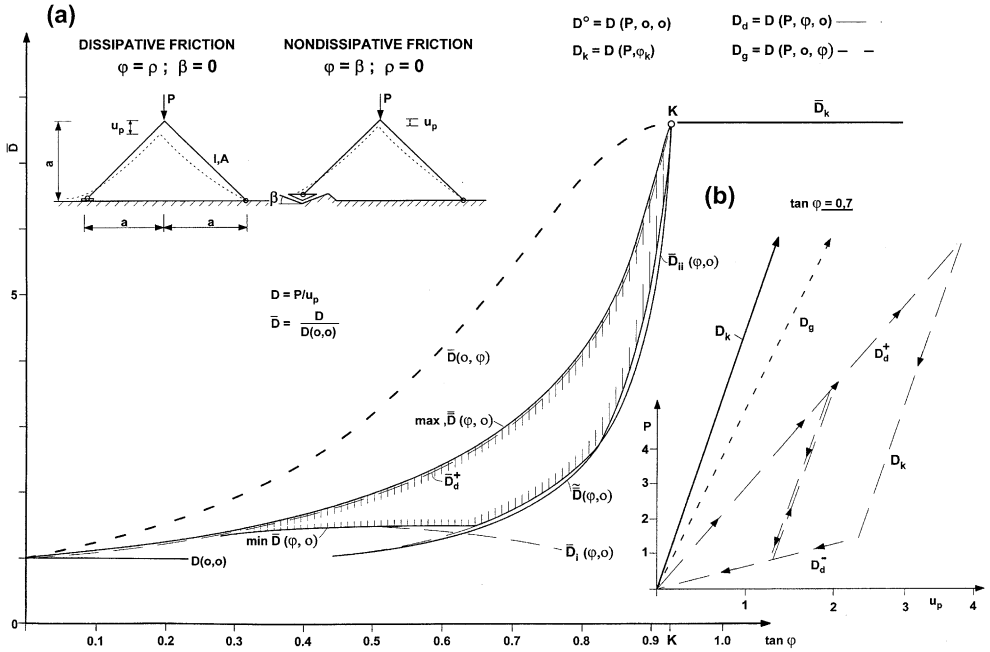

Fig. 5. Elastic frame with frictional bearing. (a) Dependence of stiffness ratio $D(0, \varphi)/D(0, 0)$ (GFA-solution) and bounds of $D(\varphi, 0)/D(0, 0)$ (DFA-solutions) on the coefficient of friction $\tan\varphi$. (b) Load P versus load-displacement u_p.

Because these stiffnesses cannot in general be uniquely determined, we can only give estimates of their upper and lower bounds. The set $\{u, \varepsilon, \gamma\}_{\rho,\beta}$ can be considered as a nondissipative varied AK-state $\{u', \varepsilon', \gamma'\}_\beta$ and $\{p(\Gamma_e), \sigma(\Omega), p(\Gamma_c)\}_{\rho,\beta}$ as a nondissipative varied AE-state $\{p^*, \sigma''(\Omega), p''(\Gamma_c)\}_\varphi$. We recall the stiffness expressions (60a) and (60b) where $D'_\varepsilon(\beta)$ is expressed by $\{u', \varepsilon', \gamma'\}_\beta$ and $D''_\sigma(\beta)$ by $\{p^*, \sigma''(\Omega), p''(\Gamma_c)\}_\varphi$. Eqs. (74a) and (74b) and the extremum principles of stiffness (Eqs. (61a) and (61b)) provide then the relations

$$\begin{aligned} &\min_{\rho\geqslant 0} D(\rho,\beta) \geqslant \inf_{u'(\beta)} D'_\varepsilon(\beta) = D(0,\beta) \\ &\max_{\rho\geqslant 0} D(\rho,\varphi-\rho) \leqslant \sup_{\sigma''(\varphi)} D''_\sigma(\varphi) = D(0,\varphi) \end{aligned} \tag{75}$$

Using the inequalities (75) in turn by varying φ, ρ, β we get the following proposition.

Proposition 6. *The following sequence of stiffnesses hold for* $0 \leqslant \rho,\ \beta \leqslant \pi/2$

$$D(0,0) \leqslant \begin{Bmatrix} \max D(\beta,0) \\ \min D(\beta,0) \end{Bmatrix} \leqslant D(0,\beta) \leqslant \begin{Bmatrix} \max D(\rho,\beta) \\ \min D(\rho,\beta) \end{Bmatrix} \leqslant D(0,\rho+\beta) \leqslant D(M) \tag{76a}$$

These inequalities express that the stiffness at load $\bar{p}^*$ increases monotonically with ρ and β towards the stiffness $D(M)$ of the monolithic structure. At constant total friction angle $\varphi = \rho + \beta$ and load p^* there holds $\min D(\varphi,0) = \inf_\beta D(\varphi-\beta,\beta) < D(\varphi-\beta,\beta) < \sup D''_\sigma(\varphi) = D(0,\varphi)$. But considering $(2W'_\varepsilon(0))^{1/2}$ as a norm of u' and using Korn's and Poincaré's inequalities, we can write

$$\min D(\varphi,0) = \inf_{u'} \frac{W'_\varepsilon(0) + \langle p(u'), u'\rangle}{\langle m^*, u'\rangle^2} \geqslant \inf_{u'} \frac{W'_\varepsilon(0)}{\langle m^*, u'\rangle^2} = \left(\sup_{u'} \frac{\langle m^*, u'\rangle}{\|u'\|}\right)^{-2} = \tilde{D}_\varepsilon(\varphi,0) \tag{76b}$$

with the restraints of Theorem 2 supplemented by the condition $|\tau(u')| \leqslant |\sigma(u')| \tan\varphi$. In this way we obtain wellposed upper and lower bounds for any stiffness $D(\varphi-\beta,\beta)$

$$\tilde{D}_\varepsilon(\varphi,0) < D(\varphi-\beta,\beta) < D(0,\varphi) \tag{76c}$$

These bounds are completely independent of the gap work $\langle p,\gamma\rangle$ at the joints but determination of the lower bound $\tilde{D}(\varphi,0)$ is in many cases cumbersome. In these cases we take into consideration also the gap work $\langle p,\gamma\rangle$ that provide still closer bounds for $D(\rho,\beta)$. Let $\{u,\sigma\}_{\rho,\beta}$ be a DFA solution for friction angles ρ, β and $\{u^b,\sigma^b\}_{0,\beta}$ be the GFA-solution for friction angles 0, β and $\{u^f,\sigma^f\}_{0,\rho+\beta}$ be the GFA-solution for friction angles 0, $\rho+\beta$. Then, if stable equilibrium at load p^* is maintained at any friction angle $\varphi \geqslant \beta$, we obtain the following bounds for the actual $\langle|\sigma|,|\gamma_t|\rangle_{\partial H}$ corresponding to friction angles ρ,β

$$\langle|\sigma^f|,|\gamma_t^f|\rangle_{\partial H} \leqslant \langle|\sigma|,|\gamma_t|\rangle_{\partial H} \leqslant \langle|\sigma^b|,|\gamma_t^b|\rangle_{\partial H} \tag{77a}$$

Indeed, the average contact sliding $|\gamma_t|$ is greatest in state $\{u^b,\sigma^b\}$, where resistance to sliding is the least, and which corresponds to the smallest total friction angle $\varphi' = \beta$. The highest resistance occurs in state $\{u^f,\sigma^f\}$ with $\varphi'' = \rho+\beta$. According to the premisses, the average $|\sigma|$ on $\Gamma_{\mu\nu}$ at the same load p^* depends mainly on γ_n and is rather independent of φ. Therefore if $\varphi > \beta$, with $\langle p,\gamma\rangle_{\partial H} = \langle|\sigma|,|\gamma_t|(\tan\varphi_\gamma - \tan\beta_\gamma)\rangle_{\partial H}$, we conclude

$$\langle|\sigma^b|,|\gamma_t^b|(\tan\varphi_{\gamma b} - \tan\beta_{\gamma b})\rangle_{\partial H} \geqslant \langle p,\gamma\rangle_{\partial H} \geqslant \langle|\sigma^f|,|\gamma_t^f|(\tan\varphi_{\gamma f} - \tan\beta_{\gamma f})\rangle_{\partial H} \geqslant 0 \tag{77b}$$

where for $\upsilon = \varphi$, β and i = b, f we used the notation $\tan\upsilon_{\gamma i} = \tan\upsilon\cos(\nabla z, \gamma_t^i)$. We label the dual pairings with σ^f, γ^f and σ^b, γ^b as $\langle p^f,\gamma^f\rangle_{\varphi,\beta}$ and $\langle p^b,\gamma^b\rangle_{\varphi,\beta}$, respectively.

A lower bound at given p^* for $\inf D(\rho,\beta)$ expressed by a state $\{u',\varepsilon',\gamma'\}_\beta$ is obtained using Eqs. (74a) and (75) and $D(\rho,\beta) = \|\bar{p}^*\|/\langle\bar{m}^*,\bar{u}\rangle_{\overline{Y}}$

$$\min D(\rho,\beta) = \inf\left(\frac{2W_\varepsilon(\rho,\beta) + \langle p',\gamma'\rangle_{\partial H}}{\langle \bar{m}^*,\bar{u}\rangle_{\bar{Y}}^2}\right) \geqslant \inf_{u'} D'_\varepsilon(\beta) + \min D^2(\rho,\beta)\inf_{u'}\left(\frac{\langle p',\gamma'\rangle_{\partial H}}{\|\bar{p}^*\|^2}\right) \tag{78a}$$

where according to Eq. (77b) $\inf\langle p,\gamma\rangle_{\partial H} \geqslant \langle p^{\rm f},\gamma^{\rm f}\rangle_{\varphi,\beta} = \langle|\sigma^{\rm f}|,|\gamma_{\rm t}^{\rm f}|(\tan\varphi_{\gamma{\rm f}} - \tan\beta_{\gamma{\rm f}})\rangle_{\partial H}$ with $\varphi_{\gamma{\rm f}} = \rho+\beta$. Introducing the notation $B(\varphi,\beta) = \|\bar{p}^*\|^2/\langle p^{\rm f},\gamma^{\rm f}\rangle_{\varphi,\beta}$ and recalling the minimum principle of Theorem 2 we obtain the inequality

$$\min D(\rho,\beta) \geqslant D(0,\beta) + (\min D(\rho,\beta))^2/B(\varphi,\beta) \tag{78b}$$

The smallest root of the above equality gives a lower bound $D(\rho,\beta)$ for $\min D(\rho,\beta)$.

$$\min D(\rho,\beta) \geqslant D_i(\rho,\beta) = \frac{2D(0,\beta)}{1 + (1 - 4D(0,\beta)/B(\varphi,\beta))^{1/2}} \tag{78c}$$

For greater friction angles ρ, β better approximations of the lower bound are obtained using work equations for the solutions $\{u,\sigma\}_{\rho,\beta}$ and $\{u^{\rm f},\sigma^{\rm f}\}_{0,\rho+\beta}$

$$\langle\bar{p}^*,\bar{u}\rangle_{\bar{Y}} = \langle\sigma,\varepsilon\rangle_H + \langle p,\gamma\rangle_{\partial H};\quad \langle\bar{p}^*,\bar{u}\rangle_{\bar{Y}} = \langle\sigma^{\rm f},\varepsilon\rangle_H + \langle p^{\rm f},\gamma\rangle_{\partial H};\quad \langle\bar{p}^*,\bar{u}^{\rm f}\rangle_{\bar{Y}} = \langle\sigma^{\rm f},\varepsilon^{\rm f}\rangle_H \tag{79a}$$

From Schwarz's inequality, with $\langle\sigma,\varepsilon\rangle = a(u,u)$, $\langle\sigma^{\rm f},\varepsilon\rangle = a(u^{\rm f},u)$, there follows $a(u^{\rm f},u)^2 = (\langle\bar{p}^*,\bar{u}\rangle_{\bar{Y}} - \langle p^{\rm f},\gamma\rangle_{\partial H})^2 \leqslant a(u^{\rm f},u^{\rm f})\cdot a(u,u) = \langle\bar{p}^*,\bar{u}^{\rm f}\rangle_{\bar{Y}}(\langle\bar{p}^*,\bar{u}\rangle_{\bar{Y}} - \langle p,\gamma\rangle_{\partial H})$. Hence $K(p^*,u) = (\langle\bar{p}^*,\bar{u}\rangle_{\bar{Y}})^2 - (2\langle p^{\rm f},\gamma\rangle_{\partial H} + \langle\bar{p}^*,\bar{u}^f\rangle_{\bar{Y}})\langle\bar{p}^*,\bar{u}\rangle_{\bar{Y}} + \langle\bar{p}^*,\bar{u}^{\rm f}\rangle_{\bar{Y}}\langle p,\gamma\rangle_{\partial H} + (\langle p^{\rm f},\gamma\rangle_{\partial H})^2 \leqslant 0$. The greatest $\langle\bar{p}^*,\bar{u}\rangle_{\bar{Y}}$ corresponds to the greatest root of the equation $K(p^*,u) = 0$

$$\sup\langle\bar{p}^*,\bar{u}\rangle_{\bar{Y}} \leqslant \langle p^*,u\rangle_{\rm ii} = \frac{\langle\bar{p}^*,\bar{u}^{\rm f}\rangle_{\bar{Y}}}{2}\left(1 + 2\frac{\langle p^{\rm f},\gamma\rangle_{\partial H}}{\langle\bar{p}^*,\bar{u}^{\rm f}\rangle_{\bar{Y}}} + \left(1 + \frac{4\langle p^{\rm f} - p,\gamma\rangle_{\partial H}}{\langle\bar{p}^*,\bar{u}^{\rm f}\rangle_{\bar{Y}}}\right)^{1/2}\right) \tag{79b}$$

This inequality is satisfied if $\langle p^{\rm f},\gamma\rangle_{\partial H} < 2\langle p^{\rm f},\gamma^{\rm f}\rangle_{\varphi,\beta} < (\langle p,\gamma\rangle_{\partial H} + \langle p^{\rm f},\gamma^{\rm f}\rangle_{\varphi,\beta})$. Hence

$$\langle p^*,u\rangle_{\rm ii} = \frac{\langle\bar{p}^*,\bar{u}^{\rm f}\rangle_{\bar{Y}}}{2}\left(1 + 4\frac{\langle p^{\rm f},\gamma^{\rm f}\rangle_{\varphi,\beta}}{\langle\bar{p}^*,\bar{u}^{\rm f}\rangle_{\bar{Y}}} + \left(1 + \frac{\langle p^{\rm f},\gamma^{\rm f}\rangle_{\varphi,\beta}}{\langle\bar{p}^*,\bar{u}^{\rm f}\rangle_{\bar{Y}}}\right)^{1/2}\right) \tag{79c}$$

Inserting $\langle p^{\rm f},\gamma^{\rm f}\rangle_{\varphi,\beta}$ and $B(\varphi,\beta)$ into Eq. (79c) we obtain finally with $\varphi = \rho+\beta$

$$\min D(\rho,\beta) > D_{\rm ii} = \frac{2D(0,\varphi)}{1 + 4D(0,\varphi)/B(\varphi,\beta) + (1 + 4D(0,\varphi)/B(\varphi,\beta))^{1/2}} \tag{79d}$$

An upper bound for $\max D(\rho,\beta)$ is established using the GFA-solution $\{u^{\rm f},\sigma^{\rm f}\}_{0,\varphi}$. From expressions (74b) and (77b) there follows with $\varphi = \rho+\beta$

$$\max D(\rho,\beta) < \frac{\|\bar{p}^*\|^2}{\inf\limits_{\rho>0} 2W_\sigma(\rho,\varphi-\rho) + \inf\limits_{\rho>0}\langle p,\gamma\rangle_{\partial H}} = \frac{\|\bar{p}^*\|^2}{2W(0,\varphi) + \langle p^{\rm f},\gamma^{\rm f}\rangle_{\varphi,\beta}} \tag{80a}$$

because $\inf\langle p,\gamma\rangle_{\varphi,\beta} \geqslant \langle p^{\rm f},\gamma^{\rm f}\rangle_{\varphi,\beta}$. Inserting $D(0,\varphi)$ and $B(\varphi,\beta)$ we obtain

$$\max D(\rho,\beta) \leqslant D_{\rm s} = \frac{D(0,\varphi)}{1 + D(0,\varphi)/B(\varphi,\beta)} \tag{80b}$$

These estimates depend only on the solu tion $\{u^{\rm f},\sigma^{\rm f}\}_{0,\varphi}$ and satisfy the limit values $D_{\rm i} = D_{\rm ii} = D(0,0)$ in frictionless case and $D_{\rm i} = D_{\rm ii} = D(0,\varphi)$ if no dissipative work occurs.

If instead of the exact values $D(0,\beta)$ and $D(0,\varphi)$ only their approximations $D''_\sigma(0,\varphi)$ and $D'_\varepsilon(0,\varphi)$ and respective $\max\langle p^{\mathrm{f}},\gamma^{\mathrm{f}}\rangle''_{\varphi,\beta}$ and $\max\langle p^{\mathrm{f}},\gamma^{\mathrm{f}}\rangle'_{\varphi,\beta}$ of Eq. (77b) are available then, recalling Eqs. (76c), (79d) and (80b), the following estimates apply

$$\max\left\{\begin{array}{c}\frac{2D''_\sigma(0,\beta)}{1+(1-4D''_\sigma(0,\beta)/B''_\sigma(\varphi,\beta))^{1/2}}\\ \tilde{D}_\varepsilon(\varphi,0)\\ \frac{2D''_\sigma(0,\varphi)}{1+4D''_\sigma(0,\varphi)/B''_\sigma(\varphi,\beta)+(1+4D''_\sigma(0,\varphi)/B''_\sigma(\varphi,\beta))^{1/2}}\end{array}\right\}\leqslant D(\rho,\beta)\leqslant\frac{2D'_\varepsilon(0,\varphi)}{1+D'_\varepsilon(0,\varphi)/B'_\varepsilon(0,\varphi)} \tag{81a}$$

where

$$\begin{aligned}&\tilde{D}_\varepsilon(\varphi,0)=\inf_{u'}(2W'_\varepsilon(\varphi,0)/\langle\bar{m},\bar{u}'\rangle^2);|\tau(u')|\leqslant|\sigma(u')|\tan\varphi\\ &B'_\varepsilon(\varphi,\beta)=|\bar{p}^*|^2/\langle|\sigma^{\mathrm{f}}(u')|,|\gamma_{\mathrm{t}}^{\prime\mathrm{f}}|(\tan\varphi_{\gamma'}-\tan\beta_{\gamma'})\rangle_{\partial H}\\ &B''_\sigma(\varphi,\beta)=|\bar{p}^*|^2\langle|\sigma''^{\mathrm{f}}|,|\gamma_{\mathrm{t}}^{\mathrm{f}}(\sigma'')|(\tan\varphi_{\gamma''}-\tan\beta_{\gamma''})\rangle_{\partial H}\end{aligned} \tag{81b}$$

Every $D(\rho,\beta)$ defines the corresponding stiffness surface $F(\varDelta,\rho,\beta)$. This coincides with the stiffness ellipsoid EM when $P\in E_{\mathrm{k}}(\rho,\beta)$, the cone of the monolithic core (Fig. 6).

Proposition 7. *If the cone of the monolithic core* $E_k(\rho,\beta)$ *exists, this cone is uniquely determined by the total friction angle* $\varphi_k=(\rho+\beta)_k: E_k(\rho,\beta)=E_k(\varphi_k); E_k(\varphi_k,0)=E_k(0,\varphi_k)$.

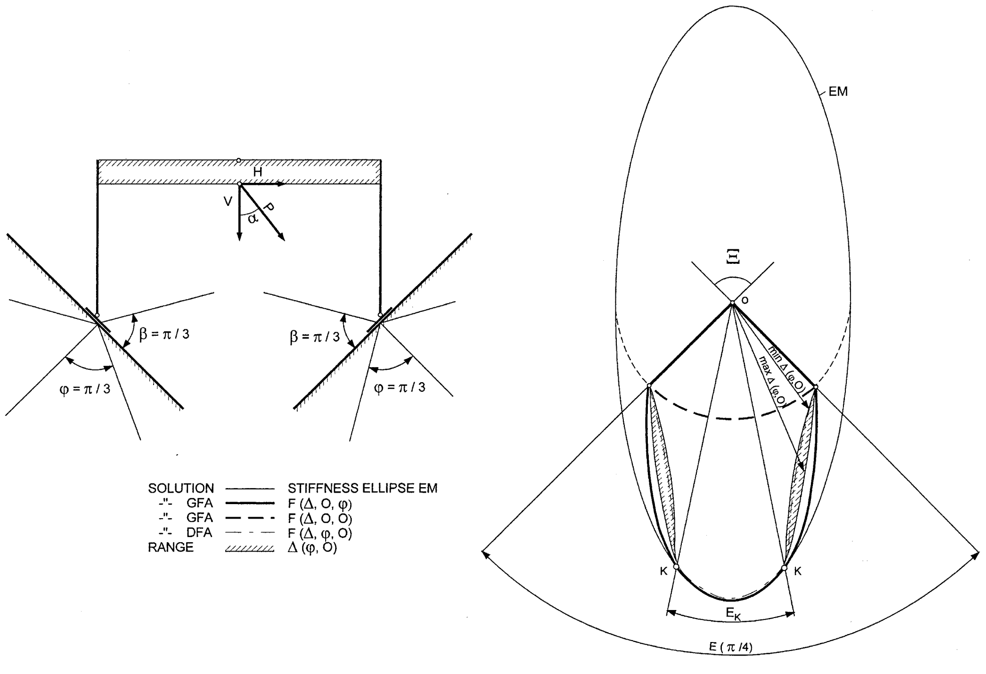

Fig. 6. Portal frame with inclined frictional bearings. The stiffness vector $\varDelta$ has a region (shaded) of indeterminateness outside the cone E_{k}, that fades away as $\varDelta$ approaches point $K\in\partial E_{\mathrm{k}}$.

Proof. Let the load $\bar{p}^*$ induce a state $\{u, \sigma\}$ of nondissipative friction ($\rho = 0, \varphi_k = \beta$), where at almost all joints $\Gamma_{\mu\nu}$ complete contact with $p(\Gamma_{\mu\nu}) \in \Phi^0(0, \varphi_k, \cdot)$ prevails and in remaining joints Γ_{ij} transitions of the $p(\Gamma_{ij})$ from $\partial\Phi(0, \varphi_k, \cdot)$ to $\Phi^0(0, \varphi_k, \cdot)$ at the edges $r_{\max}$ occur corresponding to unrestrained contact. That implies according to Proposition 5

$$\left(\frac{dD}{d\varphi}\right)_{\varphi_k} = \left(\sum_i \frac{\partial D(0,\varphi)}{\partial r_i}\frac{\partial r_i}{\partial \varphi}\right)_{r_{\max}} = 0 \tag{82}$$

This means that further increase of $\varphi > \varphi_k$ does not affect $D(0, \varphi) = D(0, \varphi_k)$ and $\Delta(0, \varphi_k)$ constitutes a generatrix ∂E_k of cone $E_k(\varphi)$. Thus ∂E_k is uniquely determined, because the solution of the nondissipative problem, if it exists, is unique. It coincides with that of the monolithic structure because $\gamma(\Gamma_k) \equiv 0$ everywhere if $p^* \in E_k(\varphi)$. The generatrix ∂E_k only depends on $\varphi_k = \rho_k + \beta_k$ and is independent of the ratio ρ_k/β_k (Fig. 5). □

Outside $E_k(\varphi)$ the surfaces $F(\Delta, \rho, \beta)$ are inside $F(\Delta, 0, \rho + \beta)$. Castigliano's and Maxwell's rules (Eqs. (64a,b)) are not valid outside $E_k(\varphi)$, but Proposition 5, concerning the stationarity of stiffness and generalized displacements with respect to the free boundaries of contact, remain valid because in the limit state there is a border $\partial\Gamma^0$ within which $p(\Gamma_c) \to \Phi^0(\varphi_k, \cdot)$. Propositions 3 and 4 remain valid with some modifications, because $\langle p, \gamma\rangle_{\partial H} \geqslant 0$: $F(\Delta, \rho, \beta)$ is not necessarily convex and the relations $U = \lambda n_F$ on $F(\Delta, 0, \beta)$ and $\langle P, U\rangle = 0$ on $\partial E(0, \beta)$ are to be replaced by $\langle \Delta, U\rangle > 0$ on $F(\Delta, \rho, \beta)$ and $\langle P, U\rangle > 0$ on $\partial E(\rho, \beta)$, respectively.

Example 3. An angle-shaped elastic frame with a frictional bearing is loaded vertically by P (Fig. 5). The stiffness $D = P/u_p$ at different friction angles are

$$\begin{aligned}
&\varphi = 0: \quad D(0,0) = \frac{\sqrt{2}\cdot 3\mathrm{EI}}{(1+c)a^3}\\
&\rho = 0: \quad D(0,\varphi) = \frac{(1+c)D(0,0)}{(1-\tan\varphi)^2 + c(1+\tan\varphi)^2}\\
&\beta = 0: \quad D(\varphi,0) = \frac{(1+c)D(0,0)}{(1-\tan\varphi) + c(1+\tan\varphi)}; \quad \tilde{D}_\varepsilon(\varphi,0) = \frac{(1+c)\left[(1-\tan\varphi)^2 + c(1+\tan\varphi)^2\right]D(0,0)}{[(1-\tan\varphi) + c(1+\tan\varphi)]^2}\\
&\varphi \geqslant \varphi_k: \quad D(M) = (1+c)^2 D(0,0)/4c; \quad \tan\varphi_k = 0.933
\end{aligned} \tag{83}$$

where $c = 3\mathrm{EJ}/2a^2\mathrm{EA} = 1/28$, EA is the compressive and EJ the bending stiffness of the struts. With $B(\varphi, 0) = D(0, 0)/((1 - c)/(1 + c) - \tan\varphi)\tan\varphi$, we obtain according to Eqs. (78c), (79d) and (80b) upper and lower bounds for $D(0, \varphi)$.

Example 4. A portal frame with a rigid beam, elastic studs and frictional bearings is loaded by $P_x = R\sin\alpha$, $P_y = R\cos\alpha$ (Fig. 6). If $\varphi = 0$, the stiffness $D(0, 0) = R/U_R = 6\mathrm{EI}/(1 + c)a^2$ is independent of α. For small values of α the stiffness surface $F(\Delta)$ coincides with the stiffness ellipse EM with major axes $(D(0, 0))^{1/2}$ and $((1 + 1/c)D(0, 0))^{1/2}$, where $c = 9\mathrm{EI}/\mathrm{EA}a^2$

$$\mathrm{EM} = \frac{(\Delta_x)^2}{D(0,0)} + \frac{c(\Delta_y)^2}{(1+c)D(0,0)} = 1 \tag{84}$$

The stiffness surfaces $F(\Delta)$ corresponding to contact sliding are determined by $\Delta_x = \sin\alpha\sqrt{D}$ and $\Delta_y = \cos\alpha\sqrt{D}$. The stiffness $D = R/U_R$ depends on ρ, β and α

$$\beta = \varphi;\quad \rho = 0;\quad D(0,\varphi)_\alpha = \frac{D(0,0)}{1-(\cos^2\alpha - \sin^2\alpha - (\cos\alpha - \sin\alpha)^2\tan\theta(1+\tan\theta))/(1+c)} \tag{85a}$$

$$\beta = 0;\quad \rho = \varphi;\quad D(\varphi,0)_\alpha = \frac{D(0,0)}{1-\cos\alpha(\cos\alpha - \sin\alpha)(1+\tan\theta)/(1+c)} \tag{85b}$$

where $\theta = \varphi - \pi/4$. The stiffness surfaces $F(\Delta,0,\varphi)$ and $F(\Delta,\varphi,0)$ are enclosed by EM and by the cone $E(R)$ of stability. They coincide with EM within the cone of the monolithic core E_k.

The indeterminateness of the stiffness vector $\Delta(\varphi,0)$ outside the cone $E_k(\varphi)$ and its disappearance at the limit of transition $K = \partial E_k \cap \mathrm{EM}$ are clearly perceptible on Figs. 5 and 6.

There are many analogies between the stiffness of elastic nonmonolithic structures and the stability of rigid body assemblages. If parts of the structures are detachable, the stable loads P^* span the interior of the cone of stability $E(\rho,\beta)$, which is convex and contains the origin. In analogy with the stiffness sequence (76a) for the cone of stability $E(\rho,\beta)$, the following inclusions hold (Parland, 1995):

$$E(0,0) \subset \begin{Bmatrix} \max E(\beta,0) \\ \min E(\beta,0) \end{Bmatrix} \subset E(0,\beta) \subset \begin{Bmatrix} \max E(\rho,\beta) \\ \min E(\rho,\beta) \end{Bmatrix} \subset E(0,\rho+\beta) \subset E(\pi/2) \tag{86}$$

where for $E(\beta,0)$ and $E(\rho,\beta)$ only some bounds can be determined. In this case every surface $F(\Delta,0,\beta)$ is contained in the corresponding cone of stability $E(0,\beta)$, and the same applies to $F(\Delta,\rho,\beta)$ and $E(\rho,\beta)$, where $\partial E(\rho,\beta)$ constitutes an osculating cone of $F(\Delta,\rho,\beta)$ at the origin. The indeterminateness of the cone of stability expressed by the set of neutral equilibrium $E_n(\rho,\beta)$ increases with the dissipativity ρ.

$$0 = E_n(0,0) = E_n(0,\rho+\beta) \subset E_n(\rho,\beta) \subset E_n(\rho+\beta,0) \tag{87a}$$

To this corresponds the extent of indeterminateness ND of the stiffnesses $D(\rho,\beta)$ expressed by the sequence of inequalities

$$0 = \mathrm{ND}(0,0) = \mathrm{ND}(0,\rho+\beta) \leqslant \mathrm{ND}(\rho,\beta) \leqslant \mathrm{ND}(\rho+\beta,0) \tag{87b}$$

because at given $\varphi = \rho + \beta$ the range of indefiniteness increases with $\tan\varphi_\gamma - \tan\beta_\gamma$.

7. Summary and conclusions

The range of Coulomb type friction angles φ, that in nonmonolithic structures warrants a unique solution, is restricted to the singles $\varphi = 0$ and $\varphi = \pi/2$. We extend this range to the whole range $[0,\pi/2]$ introducing a nondissipative geometric friction. Assuming interfaces with conforming piecewise smooth periodical asperities with maximum inclination $\tan\beta$, the gap deformation vector $\gamma(\cdot) = \{\gamma_n,\gamma_t\}^T$ and the stressvector $p(\cdot) = \{\sigma,\tau\}^T$ at the interface satisfy at contact sliding (Fig. 1) the impenetrability and friction conditions

$$\gamma_n(\cdot) \geqslant |\gamma_t(\cdot)|\tan\beta_\gamma \tag{88a}$$

$$|\tau(\cdot)| \leqslant |\sigma(\cdot)|\tan\beta_\tau;\quad \sigma(\cdot) \leqslant 0 \tag{88b}$$

Eqs. (88a) and (88b) restrict $\gamma(\cdot)$ and $p(\cdot)$ to mutually orthogonal convex cones $X(\beta,\cdot)$ and $\Phi(0,\beta,\cdot)$, respectively, where admissible $\gamma'(\cdot)$, $p''(\cdot)$ and corresponding $\gamma(\cdot)$, $p(\cdot)$ satisfy

$$\langle p'',\gamma'\rangle_{\partial H} \leqslant 0;\quad \langle p,\gamma\rangle_{\partial H} = 0;\qquad p'',p \in \Phi(0,\beta,\cdot);\quad \gamma',\gamma \in X(\beta,\cdot) \tag{89}$$

The GFA solution $\{\sigma, u\}_{0,\beta}$ corresponding to a given loading is unique, provided certain complementary conditions (Eq. (42)) are satisfied. If initial gaps $[\mathbf{r}]$ and prestress σ_0 do not occur, all rules of the monolithic structure concerning energies and their derivatives (Eqs. (64a) and (64b)) remain valid.

If the friction is dissipative with friction angle $\varphi = \rho + \beta$, where ρ stands for the dissipative Coulomb friction (Fig. 1), there holds

$$\gamma_n(\cdot) \geqslant |\gamma_t(\cdot)| \tan\beta \tag{90a}$$

$$|\tau(\cdot)| \leqslant |\sigma(\cdot)| \tan\varphi; \quad \varphi \geqslant \beta \tag{90b}$$

$p(\cdot)$ and $\gamma(\cdot)$ are contained in non-orthogonal convex cones $X(\beta, \cdot)$ and $\Phi(\rho, \beta, \cdot)$, respectively. The solution at given load p^* is not unique.

The essence of the given loading is expressed by the load perpendicular $\bar{p}^*$ with minimum norm $\|p^*\|$. The stiffness of the structure $D(\rho, \beta)$ corresponding to $\varphi = \rho + \beta$ is defined as the ratio of $\|\bar{p}^*\|$ to the load–displacement u_p in direction of $\bar{p}^*$:

(a) If the friction is nondissipative ($\rho = 0$) or $|\tau| < |\sigma| \tan\varphi$ ($\gamma_t = 0$) and $[\mathbf{r}]$, $\sigma_0 = 0$, the stiffness $D(0, \beta)$ can, using Clapeyron's equation, be expressed approximately either by an AK (admissible kinematic) state $\{u', \varepsilon', \gamma'\}$ with strain energy $W'_\varepsilon(\beta)$ as $D'_\varepsilon(\beta) = 2W'_\varepsilon(\beta)/\langle\bar{m}^*, u'\rangle^2$, or by an AE (admissible equilibrium) state $\{\sigma'', p''\}$ with stress energy $W''_\sigma(\varphi)$ as $D''_\sigma(\varphi) = \|\bar{p}^*\|^2/2W''_\sigma(\varphi)$. Now $\beta = \varphi$ and Theorem 2 provides bounds for the actual stiffness $D(0, \beta)$ at load $\bar{p}^*$

$$D''_\sigma(\beta) \leqslant D(0, \beta) \leqslant D'_\varepsilon(\beta) \tag{91a}$$

(b) If the friction is dissipative and $[\mathbf{r}]$, $\sigma_0 = 0$, only at proportional loading a consistent definition of stiffness is possible by stress and strain energies and the positive dissipative work $\langle p, \gamma\rangle_{\varphi,\beta}$ at the joints. In this case the stiffness $D(\rho, \beta)$ can be expressed alternatively by

$$D'_\varepsilon(\rho, \beta) = \frac{2W'_\varepsilon(\beta) + \langle p', \gamma'\rangle_{\partial H}}{\langle\bar{m}^*, \bar{u}\rangle^2_{\partial\bar{Y}}} \tag{91b}$$

$$D''_\sigma(\rho, \beta) = \frac{\|\bar{p}^*\|^2}{2W''_\sigma(\varphi) + \langle p'', \gamma''\rangle_{\partial H}} \tag{91c}$$

$D(\rho, \beta)$ can therefore be considered either as greater than a $D'_\varepsilon(\beta)$ corresponding to a varied AK state $\{u', \varepsilon', \gamma'\}_\beta$ or as smaller than a $D''_\sigma(\varphi)$ corresponding to a varied AE state $\{\sigma''\Omega), p''(\Gamma_c)\}_\varphi$ (Eqs. (61a)–(61c)). Thus, using the unique values of the stiffness $D(0, \beta)$ for nondissipative friction, we obtain (Eqs. (76a)–(76c)) in the general case the sequence

$$D(0,0) \leqslant \tilde{D}_\varepsilon(\varphi, 0) \leqslant \left\{ \begin{matrix} \max D(\varphi - \beta, \beta) \\ \min D(\varphi - \beta, \beta) \end{matrix} \right\} \leqslant D(0, \varphi) \leqslant D(\pi/2) \tag{92}$$

Using the contact sliding γ_t corresponding to nondissipative friction, closer bounds for the infima and suprema of the stiffness can be determined (Eq. (81a,b), Fig. 5).

If the boundaries of the contact region at given load are not fixed, the actual boundaries correspond to stationary values of the stiffness and the concerned displacements (Figs. 3 and 4). If the load is expressed by n forces P_i the stiffness vector $\{\Delta(\rho, \beta)\} = \{P\}D^{1/2}(\rho, \beta)/|P|$ determines the stiffness surface $F(\Delta, \varphi - \beta, \beta)$ that is contained in the unique stiffness surface $F(\Delta, 0, \varphi)$. This is in turn contained in the stiffness ellipsoid EM of the corresponding monolithic structure. $F(\Delta, \varphi - \beta, \beta)$ and $F(\Delta, 0, \varphi)$ coincide with EM within the cone $E_k(\varphi)$ of the monolithic core (Fig. 6). This cone with critical φ_k is uniquely determined by a GFA solution. If a structure forms an assemblage of detachable parts, the stiffness surface $F(\Delta, \rho, \beta)$ is contained in the corresponding rigid body cone of stability $E(\rho, \beta)$, that constitutes an osculating cone of $F(\Delta, \rho, \beta)$ at their common apex θ. In this case to relation (76a) concerning $D(\rho, \beta)$ and the range of its indeterminateness there corresponds a quite analogous relation (86) for $E(\rho, \beta)$.

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Appendix A. Exchange of boundary conditions of the indentation problem of an elastic strip loaded by a rigid wedge

Loading surface Γ_e; $|x| \leqslant b/2$; $y = 0$; $\mathbf{p} = \{0, p_y\}^T$; $\mathbf{u} = \{u_x, u_y\}^T$; $\mathbf{p}(x,0) \in \partial Y$; $\mathbf{u}(x,0) \in \partial Y'$; $Z = \{P_1, P_2\}$; $Z' = \{U_1, U_2\}$; $\varphi = 0$ (Fig. 7). Loading conditions $p(x,0) = p^* + p^0$; $u(x,0) = u^* + u^0$.

Nonlinear problem (Fig. 7a)

Initial gap $[r]_y = h = |x|U_2^* > 0$; $[r]_x = 0$ Rigid wedge. One degree of freedom: translation U_1^0 in direction y

Loading conditions:

$$\int_{\Gamma_e} p_y \, dx = P_1^*; \quad u_y^* = |x|U_2^* \tag{A.1a}$$

Operators $\mathrm{B} : \partial Y \to Z$; $\mathrm{C}' : Z' \to \partial Y'$

Prescribed:

$$\mathrm{B}p = \int_{\Gamma_e} p_y \, dx = P_1^* \in Z \tag{A.2a}$$

$$\int_{\Gamma_e} p_y^0 \, dx = 0 \tag{A.3a}$$

$$\mathrm{C}'U = \begin{Bmatrix} 0 \\ |x|U_2^* \end{Bmatrix} \in \partial Y' \tag{A.4a}$$

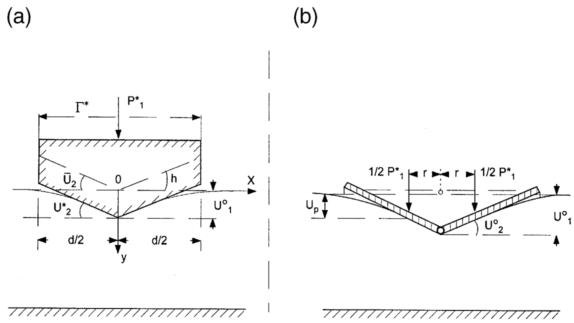

Fig. 7. Scheme of loads and displacements on the loaded surface Γ^*. (a) Generalized loads P_1^*, U_2^* prescribed. (b) P_1^* and $P_2^* = rP_1^*$ prescribed.

Complementarity requires:

$$\partial Y^0_{\mathrm{C}} \subset N(\mathrm{B});\ Z'_{\mathrm{B}} \subset N(\mathrm{C}') \quad \text{(Fig. 2)};$$

$$\int \{p^*\}^{\mathrm{T}}\{u^*\}\,\mathrm{d}x = 0;\ \int \{p^0\}^{\mathrm{T}}\{u^0\}\,\mathrm{d}x = 0$$

Hence

$$p^* = \begin{Bmatrix} 0 \\ P^*_1/b \end{Bmatrix};\ u^* = \begin{Bmatrix} 0 \\ (|x|) - b/4)U^*_2 \end{Bmatrix} \tag{A.5a}$$

$$\begin{Bmatrix} u_x \\ u_y \end{Bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & |x| - b/4 \end{bmatrix} \begin{Bmatrix} U^0_1 \\ U^*_2 \end{Bmatrix} + \begin{Bmatrix} u^0_x \\ 0 \end{Bmatrix} \tag{A.6a}$$

$$p^0_{\mathrm{C}} = \begin{Bmatrix} 0 \\ (|x| - b/4)P^0_2 \end{Bmatrix} \tag{A.7a}$$

$$\begin{Bmatrix} p_x \\ p_y \end{Bmatrix} = \begin{bmatrix} 0 & 0 \\ 1/b & |x| - b/4 \end{bmatrix} \begin{Bmatrix} P^*_1 \\ P^0_2 \end{Bmatrix} + \begin{Bmatrix} 0 \\ p^0_y \end{Bmatrix} \tag{A.8a}$$

Semilinear problem (Fig. 7b)

Initial gap $[r]_y = 0$ Wedge split into two rigid beams connected by a hinge. Two degrees of freedom: translation U^0_1 and rotation U^0_2

Loading conditions:

$$\int_{\Gamma_e} p_y\,\mathrm{d}x = P^*_1;\quad \int_{\Gamma_e} p_y|x|\,\mathrm{d}x = P^*_2 = 2(rP^*_1/2) \tag{A.1b}$$

Operators $\mathrm{B} : \partial Y \to Z$

Prescribed:

$$\mathrm{B}p = \int_{\Gamma_e} \begin{bmatrix} 0 & 1 \\ 0 & |x| \end{bmatrix} \begin{Bmatrix} p_x \\ p_y \end{Bmatrix} \mathrm{d}x = \begin{Bmatrix} P^*_1 \\ P^*_2 \end{Bmatrix} \in Z_{\mathrm{B}} \tag{A.2b}$$

$$\int_{\Gamma_e} \begin{bmatrix} 0 & 1 \\ 0 & |x| \end{bmatrix} \begin{Bmatrix} p^0_x \\ p^0_y \end{Bmatrix} \mathrm{d}x = \begin{Bmatrix} \int p^0_y\,\mathrm{d}x = 0 \\ \int p^0_y|x|\,\mathrm{d}x = 0 \end{Bmatrix} \tag{A.3b}$$

$$C' = 0 \tag{A.4b}$$

Complementarity requires:

$$u^* = 0;\ U^0 \in Z'_{\mathrm{B}};\ p^* \in \partial Y^*_{\mathrm{B}};\ p^0 \in N(\mathrm{B});$$

$$\int \{p^0\}^{\mathrm{T}}\{u^0\}\,\mathrm{d}x = 0$$

Hence

$$p^* = \begin{Bmatrix} 0 \\ \frac{12}{b^2}\left(\left(\frac{b}{3} - |x|\right)P^*_1 + \left(\frac{4|x|}{b} - 1\right)P^*_2\right) \end{Bmatrix} \tag{A.5b}$$

$$\begin{Bmatrix} u_x \\ u_y \end{Bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & |x| \end{bmatrix} \begin{Bmatrix} U^0_1 \\ -U^0_2 \end{Bmatrix} + \begin{Bmatrix} u^0_x \\ 0 \end{Bmatrix} \tag{A.6b}$$

$$u_{\mathrm{B}}^{0}=\begin{bmatrix}0 & 0\\ 1 & |x|\end{bmatrix}\begin{Bmatrix}U_1^0\\ -U_2^0\end{Bmatrix} \tag{A.7b}$$

$$\begin{Bmatrix}p_x\\ p_y\end{Bmatrix}=\frac{12}{b^2}\begin{bmatrix}0 & 0\\ b/3-|x| & 4|x|/b-1\end{bmatrix}\begin{Bmatrix}P_1^*\\ P_2^*\end{Bmatrix}+\begin{Bmatrix}0\\ p_y^0\end{Bmatrix} \tag{A.8b}$$

In both cases $\langle p_{\mathrm{C}}^{0}, u^{0}\rangle = 0$, $\langle p^{0}, u_{\mathrm{B}}^{0}\rangle = 0$ as well as $\langle p^{0}, u^{0}\rangle = 0$.

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